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COMPARING THE STOCHASTIC NONLINEAR WAVE AND HEAT EQUATIONS: A CASE STUDY

TADAHIRO OH AND MAMORU OKAMOTO

ABSTRACT. We study the two-dimensional stochastic nonlinear wave equation (SNLW) and stochastic nonlinear heat equation (SNLH) with a quadratic nonlinearity, forced by a fractional derivative (of order $\alpha > 0$) of a space-time white noise. In particular, we show that the well-posedness theory breaks at $\alpha = \frac{1}{2}$ for SNLW and at $\alpha = 1$ for SNLH. This provides a first example showing that SNLW behaves less favorably than SNLH. (i) As for SNLW, Deya (2020) essentially proved its local well-posedness for $0 < \alpha < \frac{1}{2}$. We first revisit this argument and establish multilinear smoothing of order $\frac{1}{4}$ on the second order stochastic term in the spirit of a recent work by Gubinelli, Koch, and Oh (2018). This allows us to simplify the local well-posedness argument for some range of α . On the other hand, when $\alpha \geq \frac{1}{2}$, we show that SNLW is ill-posed in the sense that the second order stochastic term is not a continuous function of time with values in spatial distributions. This shows that a standard method such as the Da Prato-Debussche trick or its variant, based on a higher order expansion, breaks down for $\alpha \geq \frac{1}{2}$. (ii) As for SNLH, we establish analogous results with a threshold given by $\alpha = 1$.

These examples show that in the case of rough noises, the existing well-posedness theory for singular stochastic PDEs breaks down before reaching the critical values ($\alpha = \frac{3}{4}$ in the wave case and $\alpha = 2$ in the heat case) predicted by the scaling analysis (due to Deng, Nahmod, and Yue (2019) in the wave case and due to Hairer (2014) in the heat case).

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1. INTRODUCTION

1.1. Singular stochastic PDEs. In this paper, we study the following stochastic nonlinear wave equation (SNLW) on $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$:

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^2 = \langle \nabla \rangle^\alpha \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \quad (x, t) \in \mathbb{T}^2 \times \mathbb{R}_+, \quad (1.1)$$

and the stochastic nonlinear heat equation (SNLH) on \mathbb{T}^2 :

$$\begin{cases} \partial_t u + (1 - \Delta)u + u^2 = \langle \nabla \rangle^\alpha \xi \\ u|_{t=0} = u_0 \end{cases} \quad (x, t) \in \mathbb{T}^2 \times \mathbb{R}_+, \quad (1.2)$$

where $\langle \nabla \rangle = \sqrt{1 - \Delta}$ and $\alpha > 0$. Namely, both equations are endowed with a quadratic nonlinearity and forced by an α -derivative of a (Gaussian) space-time white noise ξ on $\mathbb{T}^2 \times \mathbb{R}_+$.

Over the last decade, we have seen a tremendous development in the study of singular stochastic PDEs, in particular in the parabolic setting [32, 33, 28, 10, 36, 39, 12, 11, 8, 9]. Over the last few years, we have also witnessed a rapid progress in the theoretical understanding of nonlinear wave equations with singular stochastic forcing and/or rough random initial data [51, 29, 30, 31, 44, 48, 41, 43, 46, 49, 47, 42, 7]. While the regularity theory in the parabolic setting is well understood, the understanding of the solution theory in the hyperbolic/dispersive setting has been rather poor. This is due to the intricate nature of hyperbolic/dispersive problems, where case-by-case analysis is often necessary (for example, to show multilinear smoothing as in Proposition 1.4 below). Let us compare the hyperbolic and parabolic Φ_3^3 -models on the three-dimensional torus \mathbb{T}^3 as an example. In the parabolic setting [24], the standard Da Prato-Debussche trick suffices for local well-posedness, while in the wave setting, the situation is much more complicated. In [30], Gubinelli, Koch, and the first author studied the hyperbolic Φ_3^3 -model by adapting the paracontrolled calculus [28] to the hyperbolic/dispersive setting. In particular, it was essential to exploit multilinear smoothing in the construction of stochastic objects and also to introduce paracontrolled operators. While this comparison on the hyperbolic and parabolic Φ_3^3 -model shows that it may require more effort to study SNLW than SNLH, the resulting outcomes (local well-posedness on \mathbb{T}^3 with a quadratic nonlinearity forced by a space-time white noise) are essentially the same.

The main purpose of this paper is to investigate further the behavior of solutions to SNLW and SNLH and study the following question: *Does the solution theory for SNLW match up with that for SNLH?* For this purpose, we study these equations in a simpler setting of a quadratic nonlinearity on the two-dimensional torus \mathbb{T}^2 but with noises more singular than a space-time white noise (i.e. $\alpha > 0$). In this setting, we indeed provide a negative answer to the question above.

When $\alpha = 0$, the equations (1.1) and (1.2) correspond to the so-called hyperbolic Φ_2^3 -model and parabolic Φ_2^3 -model, respectively,¹ whose local well-posedness can be obtained by the standard Da Prato-Debussche trick; see [17, 29]. In this paper, we compare the behavior of solutions to these equations for more singular noises, i.e. $\alpha > 0$. We now state a “meta”-theorem.

“Theorem” 1.1. (i) *Let $0 < \alpha < \frac{1}{2}$. Then, the quadratic SNLW (1.1) is locally well-posed. When $\alpha \geq \frac{1}{2}$, the quadratic SNLW (1.1) is ill-posed in the sense the standard solution theory such as the Da Prato-Debussche trick or its variant based on a higher order expansion does not work.*

(ii) *Let $0 < \alpha < 1$. Then, the quadratic SNLH (1.2) is locally well-posed. When $\alpha \geq 1$, the quadratic SNLH (1.2) is ill-posed in the sense described above.*

For precise statements, see Theorem 1.3, Proposition 1.6, Theorem 1.10, and Proposition 1.9. Let $\alpha_* = \frac{1}{2}$ for SNLW (1.1) and $\alpha_* = 1$ for SNLH (1.2). Then, for $0 < \alpha < \alpha_*$, we prove local well-posedness of the equation via the second order expansion:²

$$u = \mathfrak{I} - \mathfrak{Y} + v. \quad (1.3)$$

Here, we adopt Hairer’s convention to denote the stochastic terms by trees; the vertex “ \cdot ” in \mathfrak{I} corresponds to the random noise $\langle \nabla \rangle^\alpha \xi$, while the edge denotes the Duhamel integral operator:³

$$\mathcal{I} = (\partial_t^2 + (1 - \Delta))^{-1} \text{ for SNLW} \quad \text{and} \quad \mathcal{I} = (\partial_t + (1 - \Delta))^{-1} \text{ for SNLH.}$$

With this notation, the stochastic convolution \mathfrak{I} and the second order stochastic term \mathfrak{Y} can be expressed as

$$\mathfrak{I} = \mathcal{I}(\langle \nabla \rangle^\alpha \xi) \quad \text{and} \quad \mathfrak{Y} = \mathcal{I}(\mathfrak{V}), \quad (1.4)$$

where \mathfrak{V} denotes a renormalized version of \mathfrak{I}^2 . See (3.2) and (5.2) for precise definitions of the stochastic convolutions. In particular, we impose $\mathfrak{I}(0) = 0$ in the wave case and $\mathfrak{I}(-\infty) = 0$ in the heat case. We then solve the fixed point problem for the residual term $v = u - \mathfrak{I} + \mathfrak{Y}$. See (1.11) and (1.17).

On the other hand, for $\alpha \geq \alpha_*$ we show that the second order term \mathfrak{Y} does *not* belong to $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ for any $T > 0$, almost surely (see Propositions 1.6 and 1.9 below). This

¹Strictly speaking, the hyperbolic Φ_2^3 -model would require a damping term $\partial_t u$ in (1.1). Since this modification does not change local well-posedness properties of the equation, we simply consider the undamped wave equation (1.1).

²As we see below, for small values of α , the first order expansion (1.5) suffices.

³In the case of SNLW, this corresponds to the forward fundamental solution to the linear Klein-Gordon equation: $\partial_t^2 u + (1 - \Delta)u = 0$.

implies⁴ that a solution u would not belong to $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ if we were to solve the equation via the second (or higher) order expansion (1.3) or the first order expansion (= the Da Prato-Debussche trick):

$$u = \mathfrak{I} + v \quad (1.5)$$

since the second order term \mathfrak{Y} appears in case-by-case analysis of the nonlinear contribution for the residual term $v = u - \mathfrak{I}$.

In Subsection 1.2, we go over details for SNLW (1.1). In Subsection 1.3, we discuss the case of SNLH (1.2).

Remark 1.2. Our main goal in this paper is to study to what extent the existing solution theory⁵ extends to handle rough noises in the context of SNLW and SNLH. For this purpose, we consider the simplest kind of nonlinearity (i.e. the quadratic nonlinearity) in (1.1) and (1.2).

There are several reasons for considering the “fractional” noise $\langle \nabla \rangle^\alpha \xi$ in (1.1) and (1.2). In studying stochastic PDEs, we often consider a noise of the form $\Phi \xi$, where Φ is a bounded operator on $L^2(\mathbb{T}^2)$. Furthermore, we often assume that Φ is Hilbert-Schmidt⁶ from $L^2(\mathbb{T}^2)$ to $H^s(\mathbb{T}^2)$. See [18, 19, 45]. It is also common to make a further assumption that a noise is spatially homogeneous. Namely, Φ is given by a convolution operator. The Bessel potential $\langle \nabla \rangle^\alpha$ is one of the simplest operator of this kind, which also allows us to tune the (spatial) regularity of the noise.

Since the work [37], fractional noises have been considered as very natural stochastic perturbation models. Stochastic PDEs with fractional noises (including $\langle \nabla \rangle^\alpha \xi$) have been studied by many researchers (see, for example, [52, 14, 54, 3, 15, 2, 34, 35, 21, 22] and the references therein). In stochastic PDEs, the first examples studied in this direction are those given by white-in-time fractional-in-space (or colored-in-space) noises [56, 18, 52, 14]. In view of the close relation of the Fourier series representation of the noise $\langle \nabla \rangle^\alpha \xi$ and the fractional-in-space noise (see Subsection 5.2 in [50]), the models (1.1) and (1.2) provide good substitutes for white-in-time fractional-in-space noises, enabling us to make an essential point without being bogged down with technical difficulties related to fractional noises. See Remark 1.13 for the case of fractional-in-time (and general fractional) noises.

1.2. Stochastic nonlinear wave equation. Stochastic nonlinear wave equations have been studied extensively in various settings; see [18, Chapter 13] for the references therein. In [29], Gubinelli, Koch, and the first author considered SNLW on \mathbb{T}^2 with an additive space-time white noise:

$$\partial_t^2 u + (1 - \Delta)u + u^k = \xi, \quad (1.6)$$

where $k \geq 2$ is an integer. The main difficulty of this problem comes from the roughness of the space-time white noise. In particular, the stochastic convolution \mathfrak{I} , solving the linear

⁴In some extreme cases, it may be possible to have $u \in C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ even if $\mathfrak{Y} \notin C([0, T]; \mathcal{D}'(\mathbb{T}^2))$, namely when the singularities of \mathfrak{Y} and v in (1.3) cancel each other. We, however, ignore such a “rare” case since it is not within the scope of the standard solution theory, (where we postulate that v is “nice”).

⁵In this paper, we restrict our attention to the solution theory based on the Da Prato-Debussche trick or its higher order variants.

⁶In the Banach space setting, we often assume that Φ is a γ -radonifying operator from $L^2(\mathbb{T}^2)$ to some Banach space B .

stochastic wave equation:

$$\partial_t^2 \mathfrak{I} + (1 - \Delta) \mathfrak{I} = \xi,$$

is not a classical function but is merely a distribution for the spatial dimension $d \geq 2$. This raises an issue in making sense of powers \mathfrak{I}^k and a fortiori of the full nonlinearity u^k in (1.6). In [29], by introducing an appropriate time-dependent renormalization, the authors proved local well-posedness of (a renormalized version of) (1.6) on \mathbb{T}^2 . See [30, 31, 48, 41, 46, 49, 47] for further work on SNLW with singular stochastic forcing. We also mention the work [21, 22] by Deya on SNLW with more singular (both in space and time) noises on bounded domains in \mathbb{R}^d and the work [55] on global well-posedness of the cubic SNLW on \mathbb{R}^2 .

We first state a local well-posedness result of the quadratic SNLW (1.1) on \mathbb{T}^2 . Given $N \in \mathbb{N}$, we define the (spatial) frequency projector π_N by

$$\pi_N u := \sum_{|n| \leq N} \widehat{u}(n) e_n, \quad (1.7)$$

where $\widehat{u}(n)$ denotes the Fourier coefficient of u and $e_n(x) = \frac{1}{2\pi} e^{in \cdot x}$ as in (2.1). We also set

$$\mathcal{H}^s(\mathbb{T}^2) = H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2). \quad (1.8)$$

Theorem 1.3. *Let $0 < \alpha < \frac{1}{2}$ and $s > \alpha$. Then, the quadratic SNLW (1.1) on \mathbb{T}^2 is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, there exists a sequence of time-dependent constants $\{\sigma_N(t)\}_{N \in \mathbb{N}}$ tending to ∞ (see (3.5) below) such that, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists an almost surely positive stopping time $T = T(\omega)$ such that the solution u_N to the following renormalized SNLW with a regularized noise:*

$$\begin{cases} \partial_t^2 u_N + (1 - \Delta) u_N + u_N^2 - \sigma_N = \langle \nabla \rangle^\alpha \pi_N \xi \\ (u_N, \partial_t u_N)|_{t=0} = (u_0, u_1) \end{cases} \quad (1.9)$$

converges almost surely to some limiting process $u \in C([0, T]; H^{-\alpha-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$.

In [22], Deya proved Theorem 1.3 on bounded domains on \mathbb{R}^2 but the same proof essentially applies on \mathbb{T}^2 .⁷ For $0 < \alpha < \frac{1}{3}$, the standard Da Prato-Debussche argument suffices to prove Theorem 1.3. Indeed, with the first order expansion (1.5), the residual term $v = u - \mathfrak{I}$ satisfies

$$\begin{aligned} \partial_t^2 v + (1 - \Delta) v &= -(v + \mathfrak{I})^2 \\ &= -v^2 - 2v\mathfrak{I} - \mathfrak{V}. \end{aligned} \quad (1.10)$$

At the second equality, we performed the Wick renormalization: $\mathfrak{I}^2 \rightsquigarrow \mathfrak{V}$. It is easy to see that \mathfrak{I} and \mathfrak{V} have regularities⁸ $-\alpha-$ and $-2\alpha-$, respectively (see Lemma 3.1 below). Then, thanks to one degree of smoothing from the wave Duhamel integral operator, we expect

⁷One may invoke the finite speed of propagation and directly apply the result in [22] to \mathbb{T}^2 . We also point out that the paper [22] handles noises with rougher temporal regularity than the space-time white noise and Theorem 1.3 is a subcase of the main result in [22].

⁸In the following, we restrict our attention to spatial regularities. Moreover, we use $a-$ (and $a+$) to denote $a - \varepsilon$ (and $a + \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. If this notation appears in an estimate, then an implicit constant is allowed to depend on $\varepsilon > 0$ (and it usually diverges as $\varepsilon \rightarrow 0$).

that v has regularity $1 - 2\alpha -$. The restriction $\alpha < \frac{1}{3}$ appears from $(1 - 2\alpha -) + (-\alpha -) > 0$ in making sense of the product $v\mathfrak{I}$ in (1.10).⁹ Then, by viewing

$$(u_0, u_1, \mathfrak{I}, \mathfrak{V})$$

as a given *enhanced data set*,¹⁰ one can easily prove local well-posedness of (1.10).

For $\frac{1}{3} \leq \alpha < \frac{1}{2}$, the argument in [22] is based on the second order expansion (1.3). In this case, the residual term $v = u - \mathfrak{I} + \mathfrak{Y}$ satisfies

$$\begin{aligned} \partial_t^2 v + (1 - \Delta)v &= -(v + \mathfrak{I} - \mathfrak{Y})^2 + \mathfrak{V} \\ &= -(v - \mathfrak{Y})^2 - 2v\mathfrak{I} + 2\mathfrak{Y}\mathfrak{I}. \end{aligned} \quad (1.11)$$

If we proceed with a “parabolic thinking”,¹¹ then we expect that \mathfrak{Y} has regularity

$$1 - 2\alpha - = 2(-\alpha -) + 1,$$

where we gain one derivative from the wave Duhamel integral operator; see (3.6). With this parabolic thinking, we see that the last product $\mathfrak{Y}\mathfrak{I}$ in (1.11) makes sense (in a deterministic manner) only for $\alpha < \frac{1}{3}$ so that $(1 - 2\alpha -) + (-\alpha -) > 0$. Nonetheless, for $\frac{1}{3} < \alpha < \frac{1}{2}$, one can use stochastic analysis to give a meaning to $\mathfrak{Y} \cdot \mathfrak{I}$ as a random distribution of regularity $-\alpha -$ (inheriting the bad regularity of \mathfrak{I}). Using the equation (1.11), we expect that v has regularity $1 - \alpha -$ and, with this regularity of v , all the terms on the right-hand side of (1.11) make sense. Then, by viewing

$$(u_0, u_1, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y}) \quad (1.12)$$

as a given enhanced data set, a standard contraction argument with the energy estimate (Lemma 2.4) yields local well-posedness of (1.11).

In view of “Theorem” 1.1, the restriction $\alpha < \frac{1}{2}$ in Theorem 1.3 is sharp. See Proposition 1.6 below. There is, however, one point that we would like to investigate in this well-posedness part. In the discussion above, we simply used a “parabolic thinking” to conclude that \mathfrak{Y} has regularity (at least) $1 - 2\alpha -$. In fact, by exploiting the explicit product structure and multilinear dispersion, we show that there is an extra smoothing for \mathfrak{Y} .

Given $N \in \mathbb{N}$, let \mathfrak{Y}_N to denote the second order term, emanating from the truncated noise $\pi_N \langle \nabla \rangle^\alpha \xi$. See (3.7) for a precise definition. We then have the following proposition.

Proposition 1.4. *Let $0 < \alpha < \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy*

$$s < s_\alpha := 1 - 2\alpha + \min\left(\alpha, \frac{1}{4}\right) = \begin{cases} 1 - \alpha, & \text{if } \alpha \leq \frac{1}{4}, \\ \frac{5}{4} - 2\alpha, & \text{if } \alpha > \frac{1}{4}. \end{cases} \quad (1.13)$$

Then, for any $T > 0$, $\{\mathfrak{Y}_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{Y} , we have

$$\mathfrak{Y} \in C([0, T]; W^{s_\alpha - \varepsilon, \infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

⁹Recall that a product of two functions is defined in general if the sum of the regularities is positive.

¹⁰Namely, once we have the pathwise regularity property of the stochastic terms \mathfrak{I} and \mathfrak{V} , we can build a continuous solution map: $(u_0, u_1, \mathfrak{I}, \mathfrak{V}) \mapsto v$ in the deterministic manner.

¹¹Namely, if we only count the regularity of each of \mathfrak{I} in \mathfrak{V} and put them together with one degree of smoothing from the wave Duhamel integral operator *without* taking into account the product structure and the oscillatory nature of the linear wave propagator.

See also Proposition 3.2 below for another instance of multilinear smoothing. In [30], such an extra smoothing property on stochastic terms via multilinear dispersion effect played an essential role in the study of the quadratic SNLW on the three-dimensional torus \mathbb{T}^3 . We believe that the multilinear smoothing in Proposition 1.4 is itself of interest since such a multilinear smoothing in the stochastic context for the wave equation is not well understood. See also Remark 1.5 below.

In our current setting, this extra smoothing does *not* improve the range of α in Theorem 1.3 since, as we will show below, the range $\alpha < \frac{1}{2}$ is sharp. Proposition 1.4, however, allows us to simplify the local well-posedness argument for the range $\frac{1}{3} \leq \alpha < \frac{5}{12}$. While the discussion above showed the Da Prato-Debussche argument to study (1.10) breaks down at $\alpha = \frac{1}{3}$, the extra smoothing in Proposition 1.4 allows us to study (1.10) at the level of the Duhamel formulation:

$$\begin{aligned} v &= S(t)(u_0, u_1) - \mathcal{I}(v^2 + 2v\mathfrak{I}) - \mathcal{I}(\mathfrak{V}) \\ &= S(t)(u_0, u_1) - \mathcal{I}(v^2 + 2v\mathfrak{I}) - \mathfrak{Y}, \end{aligned} \tag{1.14}$$

where $S(t)$ denotes the linear wave propagator defined in (2.6). Thanks to Proposition 1.4, we expect that v has regularity $\frac{5}{4} - 2\alpha -$, thus allowing us to make sense of the product $v\mathfrak{I}$ as long as $\frac{1}{3} \leq \alpha < \frac{5}{12}$, i.e. $(\frac{5}{4} - 2\alpha -) + (-\alpha -) > 0$. In this refined Da Prato-Debussche argument, the relevant enhanced data set is given by

$$(u_0, u_1, \mathfrak{I}, \mathfrak{Y}). \tag{1.15}$$

See Theorem 3.3 for a precise statement.

Alternatively, we may work with the second order expansion (1.3) and study the equation (1.11). In this case, Proposition 1.4 allows us to make sense of the product $\mathfrak{Y}\mathfrak{I}$ in the *deterministic* manner for $\alpha < \frac{5}{12}$. This in particular shows that for the range $\frac{1}{3} \leq \alpha < \frac{5}{12}$, we can solve (1.11) for $v = u - \mathfrak{I} + \mathfrak{Y}$ with a smaller enhanced data set in (1.15). Namely, when $\alpha < \frac{5}{12}$, there is no need to a priori prescribe the last term \mathfrak{Y} in (1.12). See Theorem 3.4 (i) for a precise statement.

For the range of α under consideration, i.e. $\alpha \geq \frac{1}{3}$, the extra gain of regularity in Proposition 1.4 is $\frac{1}{4}$, regardless of the value of α . When $\frac{5}{12} \leq \alpha < \frac{1}{2}$, this extra smoothing is unfortunately not sufficient to make sense of the product $\mathfrak{Y}\mathfrak{I}$ in the deterministic manner. Recalling the paraproduct decomposition (see (2.3) below), we see that the resonant product $\mathfrak{Y} := \mathfrak{Y} \odot \mathfrak{I}$ is the only issue here. Thus, for $\frac{5}{12} \leq \alpha < \frac{1}{2}$, we solve (1.11) with an enhanced data set:

$$(u_0, u_1, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y}),$$

where we use stochastic analysis to give a meaning to the problematic resonant product \mathfrak{Y} ; see Proposition 3.2.

Remark 1.5. Note that the extra smoothing is at most $\frac{1}{4}$ in Proposition 1.4, while a $\frac{1}{2}$ -smoothing was shown on \mathbb{T}^3 in [30]. This $\frac{1}{4}$ -difference in two- and three-dimensions seems to come from the effect of Lorentz transformations along null directions. The same situation appears in bilinear estimates for solutions to the linear wave equation; see, for example, Subsection 3.6 in [16]. See also Remark 4.1 for a further discussion, where (i) we show that our computation on \mathbb{T}^2 is essentially sharp and (ii) we compute the maximum possible gain

of regularity on \mathbb{T}^d , $d \geq 3$. Lastly, we point out that Proposition 1.4 states that the extra smoothing vanishes as $\alpha \rightarrow 0$.

Next, let us consider the situation for $\alpha \geq \frac{1}{2}$. In [22, Proposition 1.4], Deya showed that $\mathbb{E}[\|\mathbf{v}_N(t)\|_{H^s}^2]$ diverges for any $s \in \mathbb{R}$, when $\alpha \geq \frac{1}{2}$. This can be used to show that the Wick power \mathbf{v} is not a distribution-valued function of time when $\alpha \geq \frac{1}{2}$. The following proposition shows that the same result holds for \mathbf{y} .

Proposition 1.6. *Let $\alpha \geq \frac{1}{2}$. Then, given any $T > 0$, $\{\mathbf{y}_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ almost surely.*

We point that Proposition 1.6 is by no means to be expected from the bad behavior of \mathbf{v} for $\alpha \geq \frac{1}{2}$. For example, in the parabolic Φ_3^4 -model, it is well known that the cubic Wick power \mathbf{v} does not make sense as a distribution-valued function of time but that $\mathbf{y} = (\partial_t - \Delta)^{-1} \mathbf{v}$ belongs to $C(\mathbb{R}; \mathcal{C}^{\frac{1}{2}-}(\mathbb{T}^3))$; see [24, 40].¹² Furthermore, in Proposition 1.9 below, we prove that, for the quadratic SNLH (1.2), (i) the Wick power \mathbf{v} is not a distribution-valued function for $\alpha \geq \frac{1}{2}$ but (ii) \mathbf{y} in the heat case makes sense as a distribution-valued function for $\alpha < 1$. Therefore, we find it rather intriguing that for the wave equation, both \mathbf{v} and \mathbf{y} have the same threshold $\alpha = \frac{1}{2}$.

In the proof of Proposition 1.6, we show that each Fourier coefficient $\widehat{\mathbf{y}}_N(n, t)$ diverges almost surely for $\alpha \geq \frac{1}{2}$. See Remark 4.3. This divergence comes from the high-to-low energy transfer. Namely, the divergence comes from the nonlinear interaction of two incoming high-frequency waves resulting in a low-frequency wave.¹³ Such high-to-low energy transfer was exploited in proving ill-posedness of the deterministic nonlinear wave equations in negative Sobolev spaces; see [13, 43, 26].

Remark 1.7. (i) The proof of Proposition 1.6 also applies to \mathbb{T}^d . See Remark 4.3 (ii) for details. In particular, $\{\mathbf{y}_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^d))$ almost surely for $\alpha \geq 1 - \frac{d}{4}$.

(ii) It is interesting to note that we can prove local well-posedness of SNLW (1.1) for the entire range $0 < \alpha < \frac{1}{2}$ *without* using the paracontrolled approach as in the three-dimensional case [30].

Remark 1.8. In a recent preprint [20], Deng, Nahmod, and Yue introduced the notion of probabilistic scaling and the associated critical regularity. This is based on the observation that the Picard second iterate¹⁴ should be (at least) as smooth as a stochastic convolution (or a random linear solution in the context of the random data well-posedness theory). In

¹²A more fundamental example of this kind may be the space-time white noise ξ which does not make sense as a distribution-valued function of time, while we can define the stochastic convolution $\mathcal{I}(\xi)$ as a distribution-valued function by a limiting procedure.

¹³In (4.58), this corresponds to the interaction of two functions of spatial frequencies k and $n - k$ giving rise to an output function of spatial frequency n with $|n| \ll |k| \sim |n - k|$.

¹⁴More precisely, the Picard second iterate minus the linear solution (= the Picard first iterate). For the sake of simplicity, however, we refer to this as the Picard second iterate. For example, a Picard iteration scheme for SNLW (1.1) with the zero initial data yields the j th Picard iterates P_j , $j = 1, 2$, given by $P_1 = \mathbf{1}$ and $P_2 = \mathbf{1} - \mathcal{I}(\mathbf{y}) = \mathbf{1} - \mathbf{y}$. For simplicity, we refer to \mathbf{y} as the Picard second iterate in the following (where we also dropped the insignificant $-$ sign).

In the random data well-posedness theory for the quadratic NLW: $\partial_t^2 u + (1 - \Delta)u + u^2 = 0$ with random initial data (u_0^ω, u_1^ω) , the first two Picard iterates are given by $P_1 = S(t)(u_0^\omega, u_1^\omega)$ and $P_2 = P_1 - \mathcal{I}(P_1^2)$,

their terminology, the quadratic SNLW (1.1) on \mathbb{T}^2 is critical when $\alpha_* = \frac{3}{4}$. Proposition 1.6, however, shows that the Picard second iterate \mathfrak{Y} is not well defined for $\alpha \geq \frac{1}{2}$ in the sense that each Fourier coefficient $\widehat{\mathfrak{Y}}_N(n, t)$ diverges as $N \rightarrow \infty$.¹⁵ This in particular implies that the existing solution theory such as the Da Prato-Debussche trick or its higher order variants [4, 44]¹⁶ breaks down at $\alpha = \frac{1}{2}$ *before* reaching the critical value $\alpha_* = \frac{3}{4}$. See also Remark 4.3 for the general d -dimensional case.

We now make several remarks. (i) The discrepancy between the critical value $\alpha_* = \frac{3}{4}$ predicted by the probabilistic scaling and the actual value $\alpha = \frac{1}{2}$ for the non-existence of the Picard second iterate \mathfrak{Y} (in the limiting sense) stems from the fact that, as discussed in [20], the probabilistic scaling only takes into account several simple¹⁷ interactions (high-to-high and high-to-low) in computing a critical value. In the proof of Proposition 1.6, we make a more precise computation in proving the divergence of the Picard second iterate. (ii) As we see in the next subsection, an analogous phenomenon occurs for the quadratic SNLH (1.2) on \mathbb{T}^2 . More precisely, while the critical value predicted by the scaling analysis for (1.2) is $\alpha = 2$, the Picard second iterate \mathfrak{Y} fails to exist already at $\alpha = 1$ in the heat case. See Remark 1.12 below. In both the wave and heat cases, this pathological behavior (i.e. the divergence of the Picard second iterate and thus the breakdown of the existing solution theory) before reaching the predicted critical values seems to be closely related to the fact that we are dealing with *very* rough noises (rougher than the space-time white noise). This is in particular relevant in studying a stochastic PDE (or a deterministic PDE with random initial data) with a nonlinearity of low degree (and also in low dimensions). For example, we may expect a similar discrepancy for the nonlinear Schrödinger equation (NLS) with a quadratic nonlinearity:

$$i\partial_t u - \Delta u + \mathcal{N}(u, \bar{u}) = 0$$

with rough random initial data, where $\mathcal{N}(u, \bar{u}) = u^2$, \bar{u}^2 , or $|u|^2$ (with a proper renormalization).

1.3. Stochastic nonlinear heat equation. In this subsection, we go over the corresponding results for the quadratic SNLH (1.2) on \mathbb{T}^2 . With $\mathcal{I} = (\partial_t + (1 - \Delta))^{-1}$, let \mathfrak{I} and \mathfrak{Y} be as in (1.4) and \mathfrak{V} be the Wick renormalization of \mathfrak{I}^2 . We first state the crucial regularity result for the stochastic terms.

Proposition 1.9. (i) *For $0 < \alpha < \frac{1}{2}$ and $\varepsilon > 0$, $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}_+; C^{-2\alpha-\varepsilon}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{V} , we have*

$$\mathfrak{V} \in C(\mathbb{R}_+; C^{-2\alpha-\varepsilon}(\mathbb{T}^2))$$

almost surely. On the other hand, for $\alpha \geq \frac{1}{2}$, $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ for any $T > 0$, almost surely. Here, $\mathcal{C}^s(\mathbb{T}^2)$ denotes the Hölder-Besov space defined in (2.2).

where a proper renormalization is applied to P_1^2 . Once again, for the sake of simplicity, we refer to $P_2 - P_1 = \mathcal{I}(P_1^2) = \mathcal{I}((S(t)(u_0^\omega, u_1^\omega))^2)$ as the Picard second iterate in this discussion.

¹⁵While we work on the quadratic nonlinear wave equation (NLW) with a stochastic forcing, the same divergence result also holds for the quadratic NLW with random initial data considered in [20].

¹⁶This includes the paracontrolled approach used in [30].

¹⁷A “critical” value should be something which can be computed in advance without too much difficulty. In this sense, the simplification made in [20] in capturing main interactions seems appropriate.

(ii) For $0 < \alpha < 1$ and $\varepsilon > 0$, $\{\Upsilon_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}_+; \mathcal{C}^{2-2\alpha-\varepsilon}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by Υ , we have

$$\Upsilon \in C(\mathbb{R}_+; \mathcal{C}^{2-2\alpha-\varepsilon}(\mathbb{T}^2))$$

almost surely. On the other hand, for $\alpha \geq 1$, $\{\Upsilon_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ for any $T > 0$, almost surely.

In short, Proposition 1.9 states that \mathfrak{v} is a distribution-valued function if and only if $\alpha < \frac{1}{2}$, while Υ is a distribution-valued function if and only if $\alpha < 1$. Hence, for the range $\frac{1}{2} \leq \alpha < 1$, while $\mathfrak{v}(t)$ does not make sense as a spatial distribution, $\mathfrak{v} = (\partial_t + (1 - \Delta))\Upsilon$ makes sense as a space-time distribution. As mentioned above, such a phenomenon is already known for the parabolic Φ_3^4 -model; see [24, 40]. Proposition 1.9 exhibits sharp contrast with the situation for SNLW discussed earlier (Proposition 1.6 above), where the threshold $\alpha = \frac{1}{2}$ applies to both \mathfrak{v} and Υ .

We now state a sharp local well-posedness result for the quadratic SNLH (1.2).

Theorem 1.10. *Let $0 < \alpha < 1$ and $s > -\alpha - \varepsilon$ for sufficiently small $\varepsilon > 0$. Then, the quadratic SNLH (1.2) on \mathbb{T}^2 is locally well-posed in $\mathcal{C}^s(\mathbb{T}^2)$. More precisely, there exists a sequence of constants $\{\kappa_N\}_{N \in \mathbb{N}}$ tending to ∞ (see (5.4) below) such that, given any $u_0 \in \mathcal{C}^s(\mathbb{T}^2)$, there exists an almost surely positive stopping time $T = T(\omega)$ such that the solution u_N to the following renormalized SNLH:*

$$\begin{cases} \partial_t u_N + (1 - \Delta)u_N + u_N^2 - \kappa_N = \langle \nabla \rangle^\alpha \pi_N \xi \\ u_N|_{t=0} = u_0 \end{cases}$$

converges almost surely to some limiting process $u \in C([0, T]; \mathcal{C}^{-\alpha-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$.

In [17], Da Prato and Debussche proved Theorem 1.10 for $\alpha = 0$. The same proof based on the Da Prato-Debussche trick also applies for $0 < \alpha < \frac{2}{3}$. In this case, with the first order expansion (1.5), the residual term $v = u - \mathfrak{t}$ satisfies

$$\partial_t v + (1 - \Delta)v = -v^2 - 2v\mathfrak{t} - \mathfrak{v}, \quad (1.16)$$

where \mathfrak{t} and \mathfrak{v} have regularities $-\alpha-$ and $-2\alpha-$, respectively. Then, by repeating the analysis in the previous subsection with two degrees of smoothing coming from the heat Duhamel integral operator, v has expected regularity $2 - 2\alpha-$ and thus the restriction $\alpha < \frac{2}{3}$ appears from $(2 - 2\alpha-) + (-\alpha-) > 0$ in making sense of the product $v\mathfrak{t}$ in (1.16). Then, local well-posedness of (1.16) easily follows with an enhanced data set $(u_0, \mathfrak{t}, \Upsilon)$.

For $\frac{2}{3} \leq \alpha < 1$, the proof of Theorem 1.10 is based on the second order expansion (1.3) and proceeds exactly as in the wave case (but without any multilinear smoothing).¹⁸ In this case, the residual term $v = u - \mathfrak{t} + \Upsilon$ satisfies

$$\partial_t v + (1 - \Delta)v = -(v - \Upsilon)^2 - 2v\mathfrak{t} + 2\Upsilon\mathfrak{t}. \quad (1.17)$$

When $\alpha \geq \frac{2}{3}$, we can not make sense of the last product $\Upsilon\mathfrak{t}$ in the deterministic manner. Using stochastic analysis, we can give a meaning to $\Upsilon\mathfrak{t}$ as a distribution of regularity $-\alpha-$ for $\frac{2}{3} \leq \alpha < 1$. See Lemma 5.2. In this case, v has expected regularity of $2 - \alpha-$ and thus the restriction $\alpha < 1$ also appears in making sense of the product $v\mathfrak{t}$, namely from

¹⁸Since there is no multilinear smoothing for the heat equation, “parabolic thinking” provides a correct insight.

$(2 - \alpha -) + (-\alpha -) > 0$. Then, by applying the standard Schauder estimate, we can easily prove local well-posedness of (1.17) with an enhanced data set:

$$(u_0, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y}').$$

Remark 1.11. Let us compare the situations for SNLW (1.1) and SNLH (1.2). In this discussion, we disregard initial data. For the quadratic SNLH (1.2), the required enhanced data set consists of \mathfrak{I} and \mathfrak{Y} when $0 \leq \alpha < \frac{2}{3}$. Namely, it involves only (the powers of) the first order process \mathfrak{I} . When $\frac{2}{3} \leq \alpha < 1$, it also involves the second order and the third order processes \mathfrak{Y} and \mathfrak{Y}' . It is interesting to note that for the quadratic SNLW (1.1), thanks to the multilinear smoothing effect (Proposition 1.4), there is now an intermediate regime $\frac{1}{3} \leq \alpha < \frac{5}{12}$, where the required enhanced data set in (1.15) involves only the first and second order processes (but not the third order process). Furthermore, in this range, while the usual Da Prato-Debussche argument with (1.10) fails, the refined Da Prato-Debussche argument (1.14) at the level of the Duhamel formulation works thanks to the multilinear smoothing in Proposition 1.4.

Remark 1.12. Consider the following scaling-invariant model for the quadratic SNLH (1.2):

$$\partial_t u - \Delta u + u^2 = |\nabla|^\alpha \xi.$$

As in [39], we now apply a scaling argument to find a critical value of α . By applying the following parabolic scaling (and the associated white noise scaling for ξ):

$$\tilde{u}(x, t) = \lambda^\alpha u(\lambda x, \lambda^2 t) \quad \text{and} \quad \tilde{\xi}(x, t) = \lambda^2 \xi(\lambda x, \lambda^2 t)$$

for $\lambda > 0$, we obtain

$$\partial_t \tilde{u} - \Delta \tilde{u} + \lambda^{2-\alpha} \tilde{u}^2 = |\nabla|^\alpha \tilde{\xi}.$$

Then, by taking $\lambda \rightarrow 0$, the nonlinearity formally vanishes when $\alpha < 2$. This provides the critical value of $\alpha_* = 2$, (which agrees with the notion of local subcriticality introduced in [33]). It is very intriguing that, for the quadratic SNLH (1.2), the solution theory based on the Da Prato-Debussche trick or its higher order variants breaks down at $\alpha = 1$ *before* reaching the critical value $\alpha_* = 2$. See [34] for a similar phenomenon in the context of the KPZ equation with a fractional noise. For dispersive equations including the quadratic SNLW, the scaling analysis as above does not seem to provide any useful insight,¹⁹ unless appropriate integrability conditions are incorporated. See, for example, [25] for a discussion in the case of the stochastic nonlinear Schrödinger equation.²⁰

Remark 1.13. Lastly, we state a remark on SNLW and SNLH with a fractional-in-time noise. The space-time white noise ξ in (1.1) and (1.2) is given by a distributional time derivative of the L^2 -cylindrical Wiener process W (see (3.1) below). We may instead consider a noise $\xi^H = \partial_t W^H$ induced by a (spatially white) fractional-in-time Brownian motion W^H with the Hurst parameter $0 < H < 1$. When $H = \frac{1}{2}$, the noise ξ^H reduces to the usual space-time white noise ξ .

¹⁹For example, applying the hyperbolic scaling $(x, t) \mapsto (\lambda x, \lambda t)$, the scaling invariant version of SNLW (1.1) yields a critical value of $\alpha = \frac{5}{2}$, even higher than the heat case but the well-posedness theory for SNLW breaks down at $\alpha = \frac{1}{2}$.

²⁰In a recent preprint [20], a notion of probabilistic scaling was introduced. While the criticality associated with this notion seems to provide a good intuition for many problems, it does not provide a good prediction for the quadratic SNLW (1.1). See Remark 1.8.

We recall that the stochastic convolution $\mathcal{I}(\xi^H) = \mathcal{I}_{\text{heat}}(\xi^H)$, emanating from the fractional-in-time noise ξ^H , has (spatial) regularity $2H - 1-$. See, for example, Theorem 4 in [54]. Namely, SNLH (1.2) with the noise $\langle \nabla \rangle^\alpha \xi$ formally corresponds to the quadratic SNLH with the fractional-in-time noise ξ^H with the Hurst parameter $H = \frac{1-\alpha}{2}$ and the well-posedness result in Theorem 1.10 for $\alpha < 1$ seems to carry to the fractional-in-time noise case with $0 < H < \frac{1}{2}$. Note that the threshold value $\alpha = 1$ in Theorem 1.10 (and Proposition 1.9 (ii)) corresponds to the $H = 0$ case, which we do not discuss here. See, for example, [27] for the study of the fractional Brownian motion with $H = 0$ (which is a Gaussian process with stationary increments and logarithmic increment structure).

In the case of the wave equation, the stochastic convolution $\mathcal{I}(\xi^H) = \mathcal{I}_{\text{wave}}(\xi^H)$, emanating from the fractional-in-time noise ξ^H , has (spatial) regularity $H - \frac{1}{2}-$. See Proposition 1.2 in [21]. Thus, SNLW (1.1) formally corresponds to the quadratic SNLW with the fractional-in-time noise ξ^H with the Hurst parameter $H = \frac{1}{2} - \alpha$. In this case, local well-posedness of the quadratic SNLW with the fractional-in-time noise ξ^H is known to hold for $0 < H < \frac{1}{2}$ (corresponding to the range $0 < \alpha < \frac{1}{2}$ in Theorem 1.3). See [21, 22]. Note that the threshold value $\alpha = \frac{1}{2}$ in the wave case also corresponds to the $H = 0$ case. We also point out that in this fractional-in-time noise case, the regularities of the stochastic convolutions $\mathcal{I}_{\text{wave}}(\xi^H)$ and $\mathcal{I}_{\text{heat}}(\xi^H)$ for the wave and heat equations agree only when $H = \frac{1}{2}$.

It is also possible to consider a noise $\xi^{\vec{H}} = \partial_t W^{\vec{H}}$ coming from a space-time fractional Brownian motion $W^{\vec{H}}$ with the Hurst parameter $\vec{H} = (H_0, H_1, H_2)$, $0 < H_j < 1$, where H_0 corresponds to the temporal direction and H_1 and H_2 correspond to the two spatial directions. See, for example, [21, 22] in the wave case. In this setting, the threshold value $\alpha = \frac{1}{2}$ for SNLW (1.1) corresponds to $H_0 + H_1 + H_2 = 1$ and in this case, we expect the divergence of Y_N .²¹ In the heat case, the stochastic convolution $\mathcal{I}(\xi^{\vec{H}}) = \mathcal{I}_{\text{heat}}(\xi^{\vec{H}})$, emanating from the space-time fractional noise $\xi^{\vec{H}}$, has (spatial) regularity $2H_0 + H_1 + H_2 - 2-$. In this case, the threshold value $\alpha = 1$ for SNLH (1.2) corresponds to $2H_0 + H_1 + H_2 = 1$, at which we expect an analogous divergence of the second order process Y (in the limiting sense). We do not pursue this direction in this paper.

This paper is organized as follows. In Section 2, we introduce some notations and recall useful lemmas. In Section 3, assuming the regularity properties of the stochastic objects, we prove local well-posedness of SNLW (1.1) (Theorem 1.3). We then present details of the construction of the stochastic objects in Section 4. In particular, we prove the multilinear

²¹In the fractional noise case, a subscript N signifies that it is a stochastic process, coming from a certain approximation $W_N^{\vec{H}}$ of $W^{\vec{H}}$. See [21, 22].

In [22], the divergence of V_N is established for $H_0 + H_1 + H_2 \leq 1$. In the dispersive setting, however, it is more important to study the property of Y_N , i.e. V_N under the Duhamel integral operator since a common practice in dispersive PDEs is to make sense of a product under the Duhamel integral operator, exploiting multilinear dispersion. For example, if we consider the stochastic cubic NLS on \mathbb{T} , forced by a space-time white noise: $i\partial_t u - \partial_x^2 u + |u|^2 u = \xi$ (with a proper renormalization), then the renormalized product $\mathsf{V} = :|\mathcal{I}(\xi)|^2 \mathcal{I}(\xi):$ of the three copies of the stochastic convolution $\mathcal{I}(\xi) = \mathcal{I}_{\text{Schrödinger}}(\xi)$ does not make sense as a distribution-valued function. On the other hand, it is not difficult to see that the Picard second iterate $\mathsf{Y} = \mathcal{I}(:|\mathcal{I}(\xi)|^2 \mathcal{I}(\xi):)$ is a well defined distribution-valued function of time. The strength of Proposition 1.6 lies in showing that Y_N indeed diverges at the same threshold as V_N (which is not something we expect commonly in the study of dispersive PDEs).

smoothing for Ψ (Proposition 1.4) and divergence of Ψ (Proposition 1.6). Finally, in Section 5, we present proofs of Proposition 1.9 and Theorem 1.10.

2. BASIC LEMMAS

In this section, we introduce some notations and go over basic lemmas.

2.1. Notations. We set

$$e_n(x) := \frac{1}{2\pi} e^{in \cdot x}, \quad n \in \mathbb{Z}^2, \quad (2.1)$$

for the orthonormal Fourier basis in $L^2(\mathbb{T}^2)$. Given $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{T}^2)$ by the norm:

$$\|f\|_{H^s(\mathbb{T}^2)} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^2(\mathbb{Z}^2)},$$

where $\widehat{f}(n)$ is the Fourier coefficient of f and $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We then set $\mathcal{H}^s(\mathbb{T}^2) = H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$ as in (1.8). Similarly, given $s \in \mathbb{R}$ and $p \geq 1$, we define the L^p -based Sobolev space (Bessel potential space) $W^{s,p}(\mathbb{T}^2)$ by the norm:

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p} = \|\mathcal{F}^{-1}(\langle n \rangle^s \widehat{f}(n))\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^2) = W^{s,2}(\mathbb{T}^2)$. When we work with space-time function spaces, we use short-hand notations such as $C_T H_x^s = C([0, T]; H^s(\mathbb{T}^2))$.

For $A, B > 0$, we use $A \lesssim B$ to mean that there exists $C > 0$ such that $A \leq CB$. By $A \sim B$, we mean that $A \lesssim B$ and $B \lesssim A$. We also use a subscript to denote dependence on an external parameter; for example, $A \lesssim_\alpha B$ means $A \leq C(\alpha)B$, where the constant $C(\alpha) > 0$ depends on a parameter α .

2.2. Besov spaces and paraproduct estimates. Given $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let \mathbf{P}_j be the (non-homogeneous) Littlewood-Paley projector onto the (spatial) frequencies $\{n \in \mathbb{Z}^2 : |n| \sim 2^j\}$ such that

$$f = \sum_{j=0}^{\infty} \mathbf{P}_j f.$$

We then define the Besov spaces $B_{p,q}^s(\mathbb{T}^2)$ by the norm:

$$\|f\|_{B_{p,q}^s} = \left\| 2^{sj} \|\mathbf{P}_j f\|_{L_x^p} \right\|_{\ell_j^q(\mathbb{N}_0)}.$$

Note that $H^s(\mathbb{T}^2) = B_{2,2}^s(\mathbb{T}^2)$. We also define the Hölder-Besov space by setting

$$\mathcal{C}^s(\mathbb{T}^2) = \mathcal{B}_{\infty,\infty}^s(\mathbb{T}^2). \quad (2.2)$$

Next, we recall the following paraproduct decomposition due to Bony [5]. See [1, 28] for further details. Given two functions f and g on \mathbb{T}^2 of regularities s_1 and s_2 , respectively, we write the product fg as

$$\begin{aligned} fg &= f \otimes g + f \ominus g + f \odot g \\ &:= \sum_{j < k-2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{|j-k| \leq 2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{k < j-2} \mathbf{P}_j f \mathbf{P}_k g. \end{aligned} \quad (2.3)$$

The first term $f \otimes g$ (and the third term $f \odot g$) is called the paraproduct of g by f (the paraproduct of f by g , respectively) and it is always well defined as a distribution of

regularity $\min(s_2, s_1 + s_2)$. On the other hand, the resonant product $f \odot g$ is well defined in general only if $s_1 + s_2 > 0$.

We have the following product estimates. See [1, 38] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting).

Lemma 2.1. (i) (paraproduct and resonant product estimates) *Let $s_1, s_2 \in \mathbb{R}$ and $1 \leq p, p_1, p_2, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|f \odot g\|_{B_{p,q}^{s_2}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B_{p_2,q}^{s_2}}.$$

When $s_1 < 0$, we have

$$\|f \odot g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}.$$

When $s_1 + s_2 > 0$, we have

$$\|f \odot g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}.$$

(ii) *Let $s_1 < s_2$ and $1 \leq p, q \leq \infty$. Then, we have*

$$\|u\|_{B_{p,q}^{s_1}} \lesssim \|u\|_{W^{s_2,p}}.$$

2.3. Product estimates and discrete convolutions. Next, we recall the following product estimates. See [29] for the proof.

Lemma 2.2. *Let $0 \leq \alpha \leq 1$.*

(i) *Suppose that $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have*

$$\|\langle \nabla \rangle^\alpha (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \left(\|f\|_{L^{p_1}(\mathbb{T}^d)} \|\langle \nabla \rangle^\alpha g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle \nabla \rangle^\alpha f\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)} \right).$$

(ii) *Suppose that $1 < p, q, r < \infty$ satisfy the scaling condition: $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{\alpha}{d}$. Then, we have*

$$\|\langle \nabla \rangle^{-\alpha} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^{-\alpha} f\|_{L^p(\mathbb{T}^d)} \|\langle \nabla \rangle^\alpha g\|_{L^q(\mathbb{T}^d)}.$$

Note that while Lemma 2.2 (ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}$ in [29], the general case $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{\alpha}{d}$ follows from a straightforward modification of the proof.

We also recall the following basic lemma on a discrete convolution.

Lemma 2.3. (i) *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d.$$

Then, we have

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

(ii) *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > d$. Then, we have*

$$\sum_{\substack{n=n_1+n_2 \\ |n_1| \sim |n_2|}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

Note that we do not have the restriction $\alpha, \beta < d$ in the resonant case (ii). Lemma 2.3 follows from elementary computations. See, for example, Lemmas 4.1 and 4.2 in [40].

2.4. Linear estimates. In this subsection, we recall linear estimates for the wave and heat equations. First, we state the energy estimate for solutions to the nonhomogeneous linear wave equation \mathbb{T}^d :

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \quad (2.4)$$

By writing (2.4) in the Duhamel formulation, we have

$$u(t) = S(t)(u_0, u_1) + \mathcal{I}(F)(t), \quad (2.5)$$

where the linear wave propagator $S(t)$ is defined by

$$S(t)(u_0, u_1) = \cos(t\langle \nabla \rangle)u_0 + \frac{\sin(t\langle \nabla \rangle)}{\langle \nabla \rangle}u_1 \quad (2.6)$$

and the wave Duhamel integral operator \mathcal{I} is defined by

$$\mathcal{I}(F)(t) = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle} F(t') dt'. \quad (2.7)$$

Then, the following energy estimate follows from (2.5), (2.7), and the unitarity of the linear wave propagator $S(t)$ in $\mathcal{H}^s(\mathbb{T}^d)$.

Lemma 2.4. *Let $s \in \mathbb{R}$. Then, the solution u to (2.4) satisfies*

$$\|u\|_{L_T^\infty H_x^s} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{L_T^1 H_x^{s-1}}$$

for any $T > 0$.

In [29, 46], the authors used the Strichartz estimates to study local well-posedness of the stochastic nonlinear wave equations. Note, however, that the Strichartz estimates are not needed for proving local well-posedness of the quadratic NLW in two dimensions. More precisely, the energy estimate (Lemma 2.4), Sobolev's inequality, and a standard contraction argument yield local well-posedness of the quadratic NLW in $H^s(\mathbb{T}^2)$ for $s > 0$.

Next, we recall the Schauder estimate for the heat equation. Let $P(t) = e^{-t(1-\Delta)}$ denote the linear heat propagator defined as a Fourier multiplier operator:

$$P(t)f = \sum_{n \in \mathbb{Z}^2} e^{-t\langle n \rangle^2} \widehat{f}(n) e_n \quad (2.8)$$

for $t \geq 0$. Then, we have the following Schauder estimate on \mathbb{T}^d .

Lemma 2.5. *Let $-\infty < s_1 \leq s_2 < \infty$. Then, we have*

$$\|P(t)f\|_{C^{s_2}} \lesssim t^{\frac{s_1-s_2}{2}} \|f\|_{C^{s_1}} \quad (2.9)$$

for any $t > 0$.

The bound (2.9) on \mathbb{T}^d follows from the decay estimate for the heat kernel on \mathbb{R}^d (see Lemma 2.4 in [1]) and the Poisson summation formula to pass such a decay estimate to \mathbb{T}^d .

2.5. Tools from stochastic analysis. Lastly, we recall useful lemmas from stochastic analysis. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space (Ω, \mathcal{F}, P) , where \mathcal{F} is the σ -algebra generated by this sequence. Given $k \in \mathbb{N}_0$, we define the homogeneous Wiener chaoses \mathcal{H}_k to be the closure (under $L^2(\Omega)$) of the span of Fourier-Hermite polynomials $\prod_{n=1}^{\infty} H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^{\infty} k_n$. We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$$

for $k \in \mathbb{N}$.

We say that a stochastic process $X : \mathbb{R}_+ \rightarrow \mathcal{D}'(\mathbb{T}^d)$ is spatially homogeneous if $\{X(\cdot, t)\}_{t \in \mathbb{R}_+}$ and $\{X(x_0 + \cdot, t)\}_{t \in \mathbb{R}_+}$ have the same law for any $x_0 \in \mathbb{T}^d$. Given $h \in \mathbb{R}$, we define the difference operator δ_h by setting

$$\delta_h X(t) = X(t+h) - X(t).$$

The following lemma will be used in studying regularities of stochastic objects. For the proof, see Proposition 3.6 in [40] and Appendix in [43]. In the following, we state the result in terms of the Sobolev space $W^{s,\infty}(\mathbb{T}^d)$ but the same result holds for the Hölder-Besov space $\mathcal{C}^s(\mathbb{T}^d)$.

Lemma 2.6. *Let $\{X_N\}_{N \in \mathbb{N}}$ and X_0 be spatially homogeneous stochastic processes : $\mathbb{R}_+ \rightarrow \mathcal{D}'(\mathbb{T}^d)$. Suppose that there exists $k \in \mathbb{N}$ such that $X_N(t)$ and $X_0(t)$ belong to $\mathcal{H}_{\leq k}$ for each $t \in \mathbb{R}_+$.*

(i) *Let $t \in \mathbb{R}_+$. If there exists $s_0 \in \mathbb{R}$ such that*

$$\mathbb{E}[|\widehat{X}_0(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0} \quad (2.10)$$

for any $n \in \mathbb{Z}^d$, then we have $X_0(t) \in W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\widehat{X}_N(n, t) - \widehat{X}_M(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0} \quad (2.11)$$

for any $n \in \mathbb{Z}^d$ and $M \geq N \geq 1$, then $\{X_N(t)\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely, thus converging to some limit $X(t)$ in $W^{s,\infty}(\mathbb{T}^d)$.

(ii) *Let $T > 0$ and suppose that (i) holds on $[0, T]$. If there exists $\theta \in (0, 1)$ such that*

$$\mathbb{E}[|\delta_h \widehat{X}_0(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0+\theta} |h|^\theta, \quad (2.12)$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, and $h \in [-1, 1]$,²² then we have $X_0 \in C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\theta}{2}$, almost surely. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\delta_h \widehat{X}_N(n, t) - \delta_h \widehat{X}_M(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0+\theta} |h|^\theta, \quad (2.13)$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, $h \in [-1, 1]$, and $M \geq N \geq 1$, then $\{X_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\theta}{2}$, almost surely, thus converging to some process X in $C([0, T]; W^{s,\infty}(\mathbb{T}^d))$.

Lastly, we recall the following Wick's theorem. See Proposition I.2 in [53].

²²We impose $h \geq -t$ such that $t+h \geq 0$.

Lemma 2.7. *Let g_1, \dots, g_{2n} be (not necessarily distinct) real-valued jointly Gaussian random variables. Then, we have*

$$\mathbb{E}[g_1 \cdots g_{2n}] = \sum \prod_{k=1}^n \mathbb{E}[g_{i_k} g_{j_k}],$$

where the sum is over all partitions of $\{1, \dots, 2n\}$ into disjoint pairs (i_k, j_k) .

3. STOCHASTIC NONLINEAR WAVE EQUATION WITH ROUGH NOISE

In this section, we consider SNLW (1.1). We first state the regularity properties of the relevant stochastic terms and reformulate the problem in terms of the residual term $v = u - \mathfrak{I}$ or $v = u - \mathfrak{I} + \mathfrak{Y}$. We then present a proof of Theorem 1.3. The analysis of the stochastic terms will be presented in Section 4.

3.1. Reformulation of SNLW. Let W denote a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:

$$W(t) = \sum_{n \in \mathbb{Z}^2} \beta_n(t) e_n, \quad (3.1)$$

where $\{\beta_n\}_{n \in \mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions on a fixed probability space (Ω, \mathcal{F}, P) conditioned so that²³ $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{Z}^2$. By convention, we normalize β_n such that $\text{Var}(\beta_n(t)) = t$. Then, the stochastic convolution $\mathfrak{I} = \mathcal{I}(\langle \nabla \rangle^\alpha \xi)$ in the wave case can be formally written as

$$\mathfrak{I} = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle^{1-\alpha}} dW(t') = \sum_{n \in \mathbb{Z}^2} e_n \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle^{1-\alpha}} d\beta_n(t'). \quad (3.2)$$

We indeed construct the stochastic convolution \mathfrak{I} in (3.2) as the limit of the truncated stochastic convolution \mathfrak{I}_N defined by

$$\mathfrak{I}_N := \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} e_n \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle^{1-\alpha}} d\beta_n(t'). \quad (3.3)$$

See Lemma 3.1 below. We then define the Wick power \mathfrak{V}_N by

$$\mathfrak{V}_N := (\mathfrak{I}_N)^2 - \sigma_N, \quad (3.4)$$

where σ_N is given by

$$\begin{aligned} \sigma_N(t) &= \mathbb{E}[(\mathfrak{I}_N(x, t))^2] = \frac{1}{4\pi^2} \sum_{|n| \leq N} \int_0^t \left[\frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle^{1-\alpha}} \right]^2 dt' \\ &= \frac{1}{8\pi^2} \sum_{|n| \leq N} \left\{ \frac{t}{\langle n \rangle^{2(1-\alpha)}} - \frac{\sin(2t\langle n \rangle)}{2\langle n \rangle^{3-2\alpha}} \right\} \sim tN^{2\alpha} \end{aligned} \quad (3.5)$$

for $\alpha > 0$. We have the following regularity and convergence properties of \mathfrak{I}_N and \mathfrak{V}_N whose proofs are presented in Section 4.

²³In particular, we take β_0 to be real-valued.

Lemma 3.1. *Let $T > 0$.*

(i) *For any $\alpha \in \mathbb{R}$ and $s < -\alpha$, $\{\mathfrak{I}_N\}_{N \in \mathbb{N}}$ defined in (3.3) is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{I} , we have*

$$\mathfrak{I} \in C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

(ii) *For any $0 < \alpha < \frac{1}{2}$ and $s < -2\alpha$, $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ defined in (3.4) is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{V} , we have*

$$\mathfrak{V} \in C([0, T]; W^{-2\alpha-\varepsilon, \infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

Next, we define the second order stochastic term \mathfrak{Y} by

$$\mathfrak{Y}(t) := \mathcal{I}(\mathfrak{V})(t) = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle} \mathfrak{V}(t') dt', \quad (3.6)$$

Then, Proposition 1.4 shows that \mathfrak{Y} is a well-defined distribution and is a limit of the truncated version:

$$\mathfrak{Y}_N = \mathcal{I}(\mathfrak{V}_N), \quad (3.7)$$

provided that $0 < \alpha < \frac{1}{2}$.

Next, we give a meaning to the third order process “ $\mathfrak{Y} = \mathfrak{Y}\mathfrak{I}$ ”. As mentioned in Section 1, we need to use stochastic analysis for this purpose when $\frac{5}{12} \leq \alpha < \frac{1}{2}$. Formally write the product $\mathfrak{Y}\mathfrak{I}$ as

$$\mathfrak{Y}\mathfrak{I} = \mathfrak{Y} \otimes \mathfrak{I} + \mathfrak{Y} \ominus \mathfrak{I} + \mathfrak{Y} \odot \mathfrak{I}.$$

The paraproducts $\mathfrak{Y} \otimes \mathfrak{I}$ and $\mathfrak{Y} \odot \mathfrak{I}$ are always well defined as long as each of \mathfrak{Y} and \mathfrak{I} is well defined. Thus, we need stochastic analysis only to give a meaning to the resonant product $\mathfrak{Y} \ominus \mathfrak{I}$.

Proposition 3.2. *Let $0 < \alpha < \frac{1}{2}$ and $s < s_\alpha - \alpha$, where s_α is as in (1.13). Set $\mathfrak{Y}_N := \mathfrak{Y}_N \otimes \mathfrak{I}_N$. Then, given $T > 0$, $\{\mathfrak{Y}_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{Y} , we have*

$$\mathfrak{Y} \in C([0, T]; W^{s_\alpha - \alpha - \varepsilon, \infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

Recall from Section 1 that the standard Da Prato-Debussche trick yields local well-posedness of SNLW (1.1) for $0 < \alpha < \frac{1}{3}$. When $\frac{1}{3} \leq \alpha < \frac{5}{12}$, we use the first order expansion (1.5) and study the Duhamel formulation (1.14).

Theorem 3.3. *Let $\frac{1}{3} \leq \alpha < \frac{5}{12}$ and $s > \alpha$. Then, the equation (1.14) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists an almost surely positive stopping time $T = T(\omega)$ such that there exists a unique solution $v \in C([0, T]; H^\sigma(\mathbb{T}^2))$ to (1.14), where $\sigma > \alpha$ is sufficiently close to α . Furthermore, the solution v depends continuously on the enhanced data set:*

$$\Xi = (u_0, u_1, \mathfrak{I}, \mathfrak{Y})$$

almost surely belonging to the class:

$$\mathcal{X}_T^{s,\varepsilon} = \mathcal{H}^s(\mathbb{T}^2) \times C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \times C([0, T]; W^{s_\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \quad (3.8)$$

for some small $\varepsilon = \varepsilon(\alpha, s) > 0$. Here, s_α is as in (1.13).

When $\frac{5}{12} \leq \alpha < \frac{1}{2}$, we use the second order expansion (1.3) and study the equation (1.11) satisfied by the residual term $v = u - \mathfrak{I} + \mathfrak{Y}$. With the paraproduct decomposition (2.3), we write (1.11) as

$$\begin{cases} \partial_t^2 v + (1 - \Delta)v = -v^2 - 2v(\mathfrak{I} - \mathfrak{Y}) - \mathfrak{Y}^2 + 2(\mathfrak{Y} \otimes \mathfrak{I} + \mathfrak{Y} \circledast + \mathfrak{Y} \otimes \mathfrak{I}) \\ (v, \partial_t v)|_{t=0} = (u_0, u_1). \end{cases} \quad (3.9)$$

We now state local well-posedness of the perturbed SNLW (3.9) for the entire range $0 < \alpha < \frac{1}{2}$.

Theorem 3.4. *Let $0 < \alpha < \frac{1}{2}$ and $s > \alpha$. Then, the Cauchy problem (3.9) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists an almost surely positive stopping time $T = T(\omega)$ such that there exists a unique solution $v \in C([0, T]; H^\sigma(\mathbb{T}^2))$ to (3.9), where $\sigma \leq s$ and $\alpha < \sigma < 1 - \alpha$. Furthermore, we have the following continuous dependence statements for some small $\varepsilon = \varepsilon(\alpha, s) > 0$.*

- (i) For $0 < \alpha < \frac{5}{12}$, the solution v depends continuously on the enhanced data set:

$$\Xi = (u_0, u_1, \mathfrak{I}, \mathfrak{Y})$$

almost surely belonging to the class $\mathcal{X}_T^{s,\varepsilon}$ defined (3.8).

- (ii) For $\frac{5}{12} \leq \alpha < \frac{1}{2}$, the solution v depends continuously on the enhanced data set:

$$\Xi = (u_0, u_1, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y} \circledast) \quad (3.10)$$

almost surely belonging to the class:

$$\begin{aligned} \mathcal{Y}_T^{s,\varepsilon} &= \mathcal{H}^s(\mathbb{T}^2) \times C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \\ &\times C([0, T]; W^{s_\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \times C([0, T]; W^{s_\alpha-\alpha-\varepsilon, \infty}(\mathbb{T}^2)). \end{aligned}$$

In Subsection 3.3, we present a proof of Theorem 3.4. In view of the pathwise regularities of the relevant stochastic terms, we simply build a continuous map, sending the enhanced data set Ξ to a solution v in the deterministic manner.

Remark 3.5. If we take $\Xi = (u_0, u_1, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y} \circledast)$ as the enhanced data set, then in order to prove Theorem 3.4, it is enough to have $\mathfrak{Y} \circledast \in C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$ almost surely. Namely, we do not need to exploit the extra multilinear smoothing for $\mathfrak{Y} \circledast$. See Remark 4.2 for a further discussion. While it is possible to replace $\mathfrak{Y} \circledast$ in (3.10) by $\mathfrak{Y} \circledast := \mathfrak{Y} \otimes \mathfrak{I} + \mathfrak{Y} \circledast + \mathfrak{Y} \otimes \mathfrak{I}$, we chose not to do so in order to emphasize the fact that the resonant product $\mathfrak{Y} \circledast$ is the only term which needs to be defined a priori. (As mentioned above, given \mathfrak{Y} and \mathfrak{I} , the paraproducts $\mathfrak{Y} \otimes \mathfrak{I}$ and $\mathfrak{Y} \circledast$ are well-defined distributions.)

We point out, however, that, in Theorem 3.4 (i), the extra smoothing on \mathfrak{Y} plays an essential role in making sense of the product $\mathfrak{Y} \mathfrak{I}$ in the deterministic manner in the range $0 < \alpha < \frac{5}{12}$.

We conclude this subsection by presenting a proof of Theorem 1.3.

Proof of Theorem 1.3. We only consider the case $\frac{5}{12} \leq \alpha < \frac{1}{2}$ and $s > \alpha$. For $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ and $N \in \mathbb{N}$, we set

$$\Xi_N := (u_0, u_1, \mathfrak{I}_N, \mathfrak{Y}_N, \mathfrak{Y}_N^\bullet).$$

By Theorem 3.4 (ii), there exists a unique local-in-time solution $v_N \in C([0, T]; H^\sigma(\mathbb{T}^2))$ to (3.9) with the enhanced data set Ξ_N , where $\alpha < \sigma < 1 - \alpha$. Then, we see that

$$u_N = v_N + \mathfrak{I}_N - \mathfrak{Y}_N$$

satisfies the renormalized SNLW (1.9).

It follows from Lemma 3.1 and Propositions 1.4 and 3.2 that Ξ_N converges almost surely to Ξ in (3.10) with respect to the $\mathcal{Y}_T^{s, \varepsilon}$ -topology. In particular, Theorem 3.4 (ii) implies that the sequence $v_N = u_N - \mathfrak{I}_N + \mathfrak{Y}_N$ converges to $v \in C([0, T]; H^\sigma(\mathbb{T}^2))$. Then, we conclude that u_N converges to

$$u = v + \mathfrak{I} - \mathfrak{Y}$$

in $C([0, T]; H^{-\alpha-\varepsilon}(\mathbb{T}^2))$. □

3.2. Proof of Theorem 3.3. In the following, we study the Duhamel formulation (1.14). Let $\frac{1}{3} \leq \alpha < \frac{5}{12}$ and $0 < T \leq 1$ and fix $\varepsilon > 0$ sufficiently small. Define a map Γ by

$$\Gamma(v)(t) = S(t)(u_0, u_1) - \mathcal{I}(v^2 + 2v\mathfrak{I})(t) - \mathfrak{Y}(t)$$

where $S(t)$ and \mathcal{I} are as in (2.6) and (2.7). In the following, we take $\alpha < \sigma \leq s$.

By the energy estimate (Lemma 2.4), we have

$$\|\Gamma(v)\|_{L_T^\infty H_x^\sigma} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T \left(\|v^2\|_{L_T^\infty H_x^{\sigma-1}} + \|v\mathfrak{I}\|_{L_T^\infty H_x^{\sigma-1}} \right) + \|\mathfrak{Y}\|_{L_T^\infty H_x^\sigma}. \quad (3.11)$$

By Sobolev's inequality, we have

$$\|v^2\|_{L_T^\infty H_x^{\sigma-1}} \lesssim \|v^2\|_{L_T^\infty L_x^{\frac{2}{2-\sigma}}} = \|v\|_{L_T^\infty L_x^{\frac{4}{2-\sigma}}}^2 \lesssim \|v\|_{L_T^\infty H_x^\sigma}^2 \quad (3.12)$$

for $0 < \sigma < 1$. From Lemmas 2.2 and 3.1, we have

$$\begin{aligned} \|v\mathfrak{I}\|_{L_T^\infty H_x^{\sigma-1}} &\lesssim \|\langle \nabla \rangle^{-\alpha-\varepsilon}(v\mathfrak{I})\|_{L_T^\infty L_x^2} \\ &\lesssim \|\langle \nabla \rangle^{\alpha+\varepsilon} v\|_{L_T^\infty L_x^2} \|\langle \nabla \rangle^{-\alpha-\varepsilon} \mathfrak{I}\|_{L_T^\infty L_x^\infty} \\ &\lesssim C_\omega \|v\|_{L_T^\infty H_x^\sigma} \end{aligned} \quad (3.13)$$

for some almost surely finite constant $C_\omega > 0$, provided that $\alpha < \sigma < 1 - \alpha$.

Then, from (3.11), (3.12), (3.13), and Proposition 1.4, we obtain

$$\|\Gamma(v)\|_{L_T^\infty H_x^\sigma} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T \left(\|v\|_{L_T^\infty H_x^\sigma} + C_\omega \right)^2 + C_\omega \quad (3.14)$$

Note that we need $\sigma \leq \frac{5}{4} - 2\alpha$ in estimating the last term in (3.11). This can be guaranteed by taking $\sigma > \alpha$ sufficiently close to α as long as $\alpha < \frac{5}{12}$. Similarly, we have

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{L_T^\infty H_x^\sigma} \lesssim T \left(\|v_1\|_{L_T^\infty H_x^\sigma} + \|v_2\|_{L_T^\infty H_x^\sigma} + C_\omega \right) \|v_1 - v_2\|_{L_T^\infty H_x^\sigma}. \quad (3.15)$$

Therefore, we conclude from (3.14) and (3.15) that a standard contraction argument yields local well-posedness of (1.14). Moreover, an analogous computation shows that the solution $v \in C([0, T]; H^\sigma(\mathbb{T}^2))$ depends continuously on the enhanced data set $\Xi = (u_0, u_1, \mathfrak{I}, \mathfrak{Y})$. This completes the proof of Theorem 3.3.

3.3. Proof of Theorem 3.4. Next, we study the perturbed SNLW (3.9). Let $\alpha < \sigma \leq s$ and $0 < T \leq 1$ and fix $\varepsilon > 0$ sufficiently small. Define a map $\tilde{\Gamma}$ by

$$\tilde{\Gamma}(v)(t) = S(t)(u_0, u_1) - \mathcal{I}(v^2 + 2v(\mathfrak{t} - \mathfrak{Y}) + \mathfrak{Y}^2 - 2\mathfrak{Y}\mathfrak{Z})(t),$$

where $\mathfrak{Y}\mathfrak{Z}$ stands for

$$\mathfrak{Y}\mathfrak{Z} = \mathfrak{Y} \otimes \mathfrak{t} + \mathfrak{Y}\mathfrak{Z} + \mathfrak{Y} \otimes \mathfrak{t}. \quad (3.16)$$

For $0 < \alpha < \frac{5}{12}$, we see from (1.13) that $s_\alpha - \alpha > 0$. Hence, Proposition 1.4 and Lemma 3.1 with Lemma 2.1 imply that $\mathfrak{Y}\mathfrak{t}$ belongs to $C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$, almost surely. On the other hand, for $\frac{5}{12} \leq \alpha < \frac{1}{2}$, Proposition 3.2 implies

$$\mathfrak{Y}\mathfrak{Z} \in C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \quad (3.17)$$

almost surely.

By the energy estimate (Lemma 2.4), we have

$$\begin{aligned} \|\tilde{\Gamma}(v)\|_{L_T^\infty H_x^\sigma} &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T \left(\|v^2\|_{L_T^\infty H_x^{\sigma-1}} + \|v(\mathfrak{t} - \mathfrak{Y})\|_{L_T^\infty H_x^{\sigma-1}} \right. \\ &\quad \left. + \|\mathfrak{Y}^2\|_{L_T^\infty H_x^{\sigma-1}} + \|\mathfrak{Y}\mathfrak{Z}\|_{L_T^\infty H_x^{\sigma-1}} \right). \end{aligned} \quad (3.18)$$

Proceeding as in (3.13) with Proposition 1.4, we have

$$\begin{aligned} \|v(\mathfrak{t} - \mathfrak{Y})\|_{L_T^\infty H_x^{\sigma-1}} &\lesssim \|\langle \nabla \rangle^{-\alpha-\varepsilon}(v(\mathfrak{t} - \mathfrak{Y}))\|_{L_T^\infty L_x^2} \\ &\lesssim \|\langle \nabla \rangle^{\alpha+\varepsilon} v\|_{L_T^\infty L_x^2} \|\langle \nabla \rangle^{-\alpha-\varepsilon}(\mathfrak{t} - \mathfrak{Y})\|_{L_T^\infty L_x^\infty} \\ &\lesssim C_\omega \|v\|_{L_T^\infty H_x^\sigma} \end{aligned} \quad (3.19)$$

for some almost surely finite constant $C_\omega > 0$, provided that $\alpha < \sigma < 1 - \alpha$. From Proposition 1.4, we also have

$$\|\mathfrak{Y}^2\|_{L_T^\infty H_x^{\sigma-1}} \leq \|\mathfrak{Y}\|_{L_T^\infty L_x^\infty}^2 \leq C_\omega. \quad (3.20)$$

Putting together (3.18), (3.19), and (3.20) with (3.12) and (3.17), we obtain

$$\|\tilde{\Gamma}(v)\|_{L_T^\infty H_x^\sigma} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T \left(\|v\|_{L_T^\infty H_x^\sigma} + C_\omega \right)^2.$$

Similarly, we have

$$\|\tilde{\Gamma}(v_1) - \tilde{\Gamma}(v_2)\|_{L_T^\infty H_x^\sigma} \lesssim T \left(\|v_1\|_{L_T^\infty H_x^\sigma} + \|v_2\|_{L_T^\infty H_x^\sigma} + C_\omega \right) \|v_1 - v_2\|_{L_T^\infty H_x^\sigma}.$$

The rest follows as in the previous subsection. This completes the proof of Theorem 3.4.

4. ON THE CONSTRUCTION OF THE RELEVANT STOCHASTIC OBJECTS

In this section, we go over the construction of the stochastic terms for SNLW (1.1). As in [30], our strategy is to estimate the second moment of the Fourier coefficient and apply Lemma 2.6. In Subsection 4.1, we briefly discuss the regularity and convergence properties of \mathfrak{t} and \mathfrak{v} (Lemma 3.1). By exploiting multilinear dispersive smoothing for \mathfrak{Y} , we then present a proof of Proposition 1.4 in Subsection 4.2. In Subsection 4.3, we establish analogous multilinear smoothing for $\mathfrak{Y}\mathfrak{Z}$ (Proposition 3.2). Lastly, in Subsection 4.4, we show that, when $\alpha \geq \frac{1}{2}$, the second order stochastic term $\mathfrak{Y}(t)$ is not a spatial distribution almost surely for any $t > 0$ (Proposition 1.6).

Let $\mathfrak{r} = \mathcal{I}(\langle \nabla \rangle^\alpha \xi)$ be the stochastic convolution defined in (3.2). Given $n \in \mathbb{Z}^2$ and $0 \leq t_2 \leq t_1$, we define $\sigma_n(t_1, t_2)$ by

$$\begin{aligned} \sigma_n(t_1, t_2) &:= \mathbb{E}[\widehat{\mathfrak{r}}(n, t_1) \widehat{\mathfrak{r}}(-n, t_2)] = \int_0^{t_2} \frac{\sin((t_1 - t') \langle n \rangle)}{\langle n \rangle^{1-\alpha}} \frac{\sin((t_2 - t') \langle n \rangle)}{\langle n \rangle^{1-\alpha}} dt' \\ &= \frac{\cos((t_1 - t_2) \langle n \rangle)}{2 \langle n \rangle^{2(1-\alpha)}} t_2 + \frac{\sin((t_1 - t_2) \langle n \rangle)}{4 \langle n \rangle^{3-2\alpha}} - \frac{\sin((t_1 + t_2) \langle n \rangle)}{4 \langle n \rangle^{3-2\alpha}}. \end{aligned} \quad (4.1)$$

Then, from (3.5) and (4.1), we have

$$\sigma_N(t) = \frac{1}{4\pi^2} \sum_{|n| \leq N} \sigma_n(t, t). \quad (4.2)$$

Moreover, from Wick's theorem (Lemma 2.7), we have

$$\mathbb{E}\left[\left(|\widehat{\mathfrak{r}}(n_1, t_1)|^2 - \sigma_{n_1}(t_1, t_1)\right)\left(|\widehat{\mathfrak{r}}(n_2, t_2)|^2 - \sigma_{n_2}(t_2, t_2)\right)\right] = \mathbf{1}_{n_1 = \pm n_2} \cdot \sigma_{n_1}^2(t_1, t_2). \quad (4.3)$$

In the following, we fix $T > 0$.

4.1. Proof of Lemma 3.1. (i) From (4.1), we have

$$\mathbb{E}[|\widehat{\mathfrak{r}}_N(n, t)|^2] = \sigma_n(t, t) \lesssim_T \langle n \rangle^{-2+2\alpha} \quad (4.4)$$

for any $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$. Also, by the mean value theorem and an interpolation argument as in [30], we have

$$\mathbb{E}[|\widehat{\mathfrak{r}}_N(n, t_1) - \widehat{\mathfrak{r}}_N(n, t_2)|^2] \lesssim_T \langle n \rangle^{-2(1-\alpha)+\theta} |t_1 - t_2|^\theta$$

for any $\theta \in [0, 1]$, $n \in \mathbb{Z}^2$, and $0 \leq t_2 \leq t_1 \leq T$ with $t_1 - t_2 \leq 1$, uniformly in $N \in \mathbb{N}$. Hence, from Lemma 2.6, we conclude that $\mathfrak{r}_N \in C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$ for any $\varepsilon > 0$, almost surely. Moreover, a slight modification of the argument yields that $\{\mathfrak{r}_N\}_{N \in \mathbb{N}}$ is almost surely a Cauchy sequence in $C([0, T]; W^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$, thus converging to some limit \mathfrak{r} . Since the required modification is exactly the same as in [30], we omit the details here.

In the remaining part of this section, we only establish the estimate (2.10) in Lemma 2.6 for each of \mathfrak{v}_N , \mathfrak{y}_N , and \mathfrak{y}_N^\bullet , uniformly in $N \in \mathbb{N}$. The time difference estimate (2.12) and the convergence claim follow from a straightforward modification as in [30].

(ii) Next, we study the Wick power \mathfrak{v}_N . In view of Lemma 2.6 and the comment above, it suffices to prove

$$\mathbb{E}[|\widehat{\mathfrak{v}}_N(n, t)|^2] \lesssim_T \langle n \rangle^{-2+4\alpha} \quad (4.5)$$

for $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$. From (3.4) and (4.2), we have

$$\begin{aligned} \widehat{\mathfrak{v}}_N(n, t) &= \widehat{\mathfrak{r}}_N^2(n, t) - \mathbf{1}_{n=0} \cdot 2\pi \sigma_N(t) \\ &= \frac{1}{2\pi} \sum_{\substack{n=n_1+n_2 \\ |n_1|, |n_2| \leq N}} \left(\widehat{\mathfrak{r}}(n_1, t) \widehat{\mathfrak{r}}(n_2, t) - \mathbf{1}_{n=0} \cdot \sigma_{n_1}(t, t) \right) \end{aligned}$$

and thus we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathbf{V}}_N(n, t)|^2] &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2 \\ |n_1|, |n_2| \leq N}} \sum_{\substack{n=n'_1+n'_2 \\ |n'_1|, |n'_2| \leq N}} \mathbb{E} \left[\left(\widehat{\mathbf{f}}(n_1, t) \widehat{\mathbf{f}}(n_2, t) - \mathbf{1}_{n=0} \cdot \sigma_{n_1}(t, t) \right) \right. \\ &\quad \left. \times \overline{\left(\widehat{\mathbf{f}}(n'_1, t) \widehat{\mathbf{f}}(n'_2, t) - \mathbf{1}_{n=0} \cdot \sigma_{n'_1}(t, t) \right)} \right]. \end{aligned} \quad (4.6)$$

In order to have non-zero contribution in (4.6), we must have $n_1 = n'_1$ and $n_2 = n'_2$ up to permutation.

By Wick's theorem (Lemma 2.7), we have

$$\mathbb{E}[|\widehat{\mathbf{f}}(n, t)|^4] = 2\sigma_n^2(t, t). \quad (4.7)$$

Then, for $n = 0$, it follows from (4.6), (4.3), and (4.7) that

$$\begin{aligned} \mathbb{E}[|\widehat{\mathbf{V}}_N(0, t)|^2] &\lesssim \sum_{k \in \mathbb{Z}^2} \mathbb{E} \left[\left(|\widehat{\mathbf{f}}(k, t)|^2 - \sigma_k(t, t) \right)^2 \right] = \sum_{k \in \mathbb{Z}^2} \left(\mathbb{E}[|\widehat{\mathbf{f}}(k, t)|^4] - \sigma_k^2(t, t) \right) \\ &= \sum_{k \in \mathbb{Z}^2} \sigma_k^2(t, t) \lesssim_T \sum_{k \in \mathbb{Z}^2} \frac{1}{\langle k \rangle^{4(1-\alpha)}} < \infty, \end{aligned} \quad (4.8)$$

provided that $\alpha < \frac{1}{2}$. Similarly, for $n \neq 0$, we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathbf{V}}_N(n, t)|^2] &= \frac{1}{2\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \mathbb{E} \left[|\widehat{\mathbf{f}}_N(n_1, t)|^2 |\widehat{\mathbf{f}}_N(n_2, t)|^2 \right] + \frac{1}{4\pi^2} \cdot \mathbf{1}_{n \in 2\mathbb{Z}^2 \setminus \{0\}} \mathbb{E} \left[|\widehat{\mathbf{f}}_N\left(\frac{n}{2}, t\right)|^4 \right] \\ &= \frac{1}{2\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1+n_2 \neq 0 \\ |n_1|, |n_2| \leq N}} \sigma_{n_1}(t, t) \sigma_{n_2}(t, t) \\ &\lesssim_T \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \lesssim \langle n \rangle^{-2+4\alpha}, \end{aligned} \quad (4.9)$$

provided that $0 < \alpha < \frac{1}{2}$. In the last inequality, we used Lemma 2.3. This proves (4.5).

4.2. Proof of Proposition 1.4. Let $0 < \alpha < \frac{1}{2}$ and let s_α be as in (1.13). In view of Lemma 2.6, it suffices to show

$$\mathbb{E}[|\widehat{\mathbf{Y}}_N(n, t)|^2] \lesssim_T \langle n \rangle^{-2-2s_\alpha} \quad (4.10)$$

for any $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$. Our argument follows closely to that in the proof of Proposition 1.6 in [30] up to Case 2 below, where our argument diverges. We, however, present details for readers' convenience. See also Remark 4.1 below.

By the definition (3.6), we have

$$\mathbb{E}[|\widehat{\mathbf{Y}}_N(n, t)|^2] = \int_0^t \frac{\sin((t-t_1)\langle n \rangle)}{\langle n \rangle} \int_0^t \frac{\sin((t-t_2)\langle n \rangle)}{\langle n \rangle} \mathbb{E} \left[\widehat{\mathbf{V}}_N(n, t_1) \overline{\widehat{\mathbf{V}}_N(n, t_2)} \right] dt_2 dt_1. \quad (4.11)$$

Let us first consider the case $n = 0$. It follows from (4.11) and (4.6) that

$$\begin{aligned} \mathbb{E}[|\hat{\mathbf{Y}}_N(0, t)|^2] &= \frac{1}{4\pi^2} \int_0^t \sin(t - t_1) \int_0^t \sin(t - t_2) \\ &\quad \times \sum_{\substack{k_1, k_2 \in \mathbb{Z}^2 \\ |k_1|, |k_2| \leq N}} \mathbb{E}\left[\left(|\hat{\mathbf{r}}(k_1, t_1)|^2 - \sigma_{k_1}(t_1, t_1)\right)\left(|\hat{\mathbf{r}}(k_2, t_2)|^2 - \sigma_{k_2}(t_2, t_2)\right)\right] dt_2 dt_1. \end{aligned}$$

By symmetry, (4.3), and (4.1), we obtain

$$\mathbb{E}[|\hat{\mathbf{Y}}_N(0, t)|^2] \lesssim \int_0^t \int_0^{t_1} \sum_{k \in \mathbb{Z}^2} \sigma_k^2(t_1, t_2) dt_2 dt_1 \lesssim_T \sum_{k \in \mathbb{Z}^2} \frac{1}{\langle k \rangle^{4(1-\alpha)}} < \infty,$$

provided that $\alpha < \frac{1}{2}$. This proves (4.10) when $n = 0$.

In the following, we consider the case $n \neq 0$. With (4.6) and proceeding as in (4.9), we have

$$\begin{aligned} \mathbb{E}[|\hat{\mathbf{Y}}_N(n, t)|^2] &= \frac{1}{\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \int_0^t \frac{\sin((t - t_1)\langle n \rangle)}{\langle n \rangle} \\ &\quad \times \int_0^{t_1} \frac{\sin((t - t_2)\langle n \rangle)}{\langle n \rangle} \sigma_{n_1}(t_1, t_2) \sigma_{n_2}(t_1, t_2) dt_2 dt_1 \\ &\quad + \frac{1}{2\pi^2} \cdot \mathbf{1}_{n \in 2\mathbb{Z}^2 \setminus \{0\}} \int_0^t \frac{\sin((t - t_1)\langle n \rangle)}{\langle n \rangle} \int_0^{t_1} \frac{\sin((t - t_2)\langle n \rangle)}{\langle n \rangle} \\ &\quad \times \mathbb{E}\left[\hat{\mathbf{r}}_N\left(\frac{n}{2}, t_1\right)^2 \overline{\hat{\mathbf{r}}_N\left(\frac{n}{2}, t_2\right)^2}\right] dt_2 dt_1 \\ &=: \text{I}(n, t) + \text{II}(n, t), \end{aligned} \tag{4.12}$$

where $\text{II}(n, t)$ denotes the contribution from $n_1 = n_2 = n'_1 = n'_2 = \frac{n}{2}$.

We first estimate the second term $\text{II}(n, t)$ in (4.12). By Wick's theorem (Lemma 2.7) with (4.1), we have

$$\left| \mathbb{E}\left[\hat{\mathbf{r}}_N\left(\frac{n}{2}, t_1\right)^2 \overline{\hat{\mathbf{r}}_N\left(\frac{n}{2}, t_2\right)^2}\right] \right| \lesssim_T \langle n \rangle^{-4(1-\alpha)}$$

under $0 \leq t_2 \leq t_1 \leq t \leq T$. Hence, from (4.12) with (1.13), we conclude that

$$|\text{II}(n, t)| \lesssim_T \langle n \rangle^{-6+4\alpha} \leq \langle n \rangle^{-2-2s_\alpha}, \tag{4.13}$$

verifying (4.10).

Next, we estimate $\text{I}(n, t)$ in (4.12). As in [30], we have

$$\begin{aligned} \text{I}(n, t) &= -\frac{1}{4\pi^2} \sum_{k_1, k_2 \in \{1, 2\}} \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} \frac{\varepsilon_1 \varepsilon_2 e^{i(\varepsilon_1 + \varepsilon_2)t\langle n \rangle}}{\langle n \rangle^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \int_0^t e^{-i\varepsilon_1 t_1 \langle n \rangle} \\ &\quad \times \int_0^{t_1} e^{-i\varepsilon_2 t_2 \langle n \rangle} \prod_{j=1}^2 \sigma_{n_j}^{(k_j)}(t_1, t_2) dt_2 dt_1 =: \sum_{k_1, k_2 \in \{1, 2\}} \text{I}^{(k_1, k_2)}(n, t), \end{aligned} \tag{4.14}$$

where $\sigma_n^{(1)}(t_1, t_2)$ and $\sigma_n^{(2)}(t_1, t_2)$ are defined by

$$\sigma_n^{(1)}(t_1, t_2) := \frac{\cos((t_1 - t_2)\langle n \rangle)}{2\langle n \rangle^{2(1-\alpha)}} t_2, \quad (4.15)$$

$$\sigma_n^{(2)}(t_1, t_2) := \frac{\sin((t_1 - t_2)\langle n \rangle)}{4\langle n \rangle^{3-2\alpha}} - \frac{\sin((t_1 + t_2)\langle n \rangle)}{4\langle n \rangle^{3-2\alpha}} \quad (4.16)$$

such that $\sigma_n(t_1, t_2) = \sigma_n^{(1)}(t_1, t_2) + \sigma_n^{(2)}(t_1, t_2)$.

By Lemma 2.3, the contribution to $I(n, t)$ in (4.14) from $(k_1, k_2) \neq (1, 1)$ can be estimated by

$$\frac{1}{\langle n \rangle^2} \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{3-2\alpha}} \lesssim \langle n \rangle^{-2-2(1-\alpha)+}$$

for $0 < \alpha < \frac{1}{2}$, verifying (4.10). Hence, we focus on estimating $I(n, t)$ coming from $(k_1, k_2) = (1, 1)$:

$$\begin{aligned} I^{(1,1)}(n, t) &:= -\frac{1}{64\pi^2} \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \frac{\varepsilon_1 \varepsilon_2 e^{i(\varepsilon_1 + \varepsilon_2)t\langle n \rangle}}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \\ &\quad \times \int_0^t e^{-it_1 \kappa_1(\bar{n})} \int_0^{t_1} t_2^2 e^{-it_2 \kappa_2(\bar{n})} dt_2 dt_1, \end{aligned} \quad (4.17)$$

where $\kappa_1(\bar{n})$ and $\kappa_2(\bar{n})$ are defined by

$$\kappa_1(\bar{n}) = \varepsilon_1 \langle n \rangle - \varepsilon_3 \langle n_1 \rangle - \varepsilon_4 \langle n_2 \rangle \quad \text{and} \quad \kappa_2(\bar{n}) = \varepsilon_2 \langle n \rangle + \varepsilon_3 \langle n_1 \rangle + \varepsilon_4 \langle n_2 \rangle. \quad (4.18)$$

When $|n| \lesssim 1$, it follows from Lemma 2.3 that $|I^{(1,1)}(n, t)| \lesssim_T 1$ for $0 < \alpha < \frac{1}{2}$. Hence, we assume $|n| \gg 1$. As in [30], we must carefully estimate $I^{(1,1)}(n, t)$, depending on $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, by exploiting either (i) the dispersion (= oscillation) or (ii) smallness of the measure of the relevant frequency set.

Fix our choice of $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and denote by $I_{\bar{\varepsilon}}^{(1,1)}(n, t)$ the associated contribution to $I^{(1,1)}(n, t)$. By switching the order of integration and first integrating in t_1 , we have

$$\begin{aligned} &\left| \int_0^t e^{-it_1 \kappa_1(\bar{n})} \int_0^{t_1} t_2^2 e^{-it_2 \kappa_2(\bar{n})} dt_2 dt_1 \right| \\ &= \left| \int_0^t t_2^2 e^{-it_2 \kappa_2(\bar{n})} \frac{e^{-it_2 \kappa_1(\bar{n})} - e^{-it_2 \kappa_1(\bar{n})}}{-i\kappa_1(\bar{n})} dt_2 \right| \lesssim_T (1 + |\kappa_1(\bar{n})|)^{-1}. \end{aligned} \quad (4.19)$$

From (4.17) and (4.19), we have

$$|I_{\bar{\varepsilon}}^{(1,1)}(n, t)| \lesssim_T \sum_{n=n_1+n_2} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_1(\bar{n})|)}. \quad (4.20)$$

In the following, we assume $|n_1| \geq |n_2|$ without loss of generality. Under $n = n_1 + n_2$ we then have

$$\langle n_1 \rangle \sim \langle n \rangle + \langle n_2 \rangle. \quad (4.21)$$

When $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \mp 1, \mp 1)$ or $(\pm 1, \mp 1, \pm 1)$, we have $|\kappa_1(\bar{n})| \geq \langle n \rangle$. Then, the desired bound (4.10) follows from (4.20) and Lemma 2.3.

Next, we consider the case $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \pm 1, \mp 1)$. In this case, we have $|\kappa_1(\bar{n})| = \langle n \rangle + \langle n_2 \rangle - \langle n_1 \rangle$. By (4.20), the contribution to $I_{\varepsilon}^{(1,1)}(n, t)$ from $n_2 = 0$ is estimated by

$$\frac{1}{\langle n \rangle^2 \langle n \rangle^{2(1-\alpha)}} = \langle n \rangle^{-4+2\alpha} \leq \langle n \rangle^{-2-2s_\alpha},$$

satisfying (4.10). Hence, we assume $n_2 \neq 0$ in the following. By viewing n_1 as a vector based at n_2 , we see that three vectors n , n_1 , and n_2 form a triangle. Hence, it follows from the law of cosines that

$$|n|^2 + |n_2|^2 - |n_1|^2 = 2|n||n_2| \cos(\angle(n, n_2)), \quad (4.22)$$

where $\angle(n, n_2)$ denotes the angle between n and n_2 . Then, from (4.21) and (4.22), we have

$$\begin{aligned} |\kappa_1(\bar{n})| &= \frac{(\langle n \rangle + \langle n_2 \rangle)^2 - \langle n_1 \rangle^2}{\langle n \rangle + \langle n_2 \rangle + \langle n_1 \rangle} = \frac{2\langle n \rangle \langle n_2 \rangle + |n|^2 + |n_2|^2 - |n_1|^2 + 1}{\langle n \rangle + \langle n_2 \rangle + \langle n_1 \rangle} \\ &\gtrsim \frac{1 + |n||n_2|(1 - \cos \theta)}{\langle n_1 \rangle}, \end{aligned} \quad (4.23)$$

where $\theta = \angle(n_2, -n) \in [0, \pi]$ is the angle between n_2 and $-n$.

Using (4.20), the contribution to $I_{\varepsilon}^{(1,1)}(n, t)$ from $n_2 \neq 0$ is estimated by

$$\begin{aligned} &\lesssim \sum_{\substack{n=n_1+n_2 \\ 1 \leq |n_2| \leq |n_1| \\ 1 - \cos \theta \gtrsim 1}} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_1(\bar{n})|)} \\ &\quad + \sum_{\substack{n=n_1+n_2 \\ 1 \leq |n_2| \leq |n_1| \\ 1 - \cos \theta \ll 1}} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_1(\bar{n})|)} \\ &=: I_{\varepsilon,1}^{(1,1)}(n, t) + I_{\varepsilon,2}^{(1,1)}(n, t). \end{aligned} \quad (4.24)$$

In the following, we separately estimate $I_{\varepsilon,1}^{(1,1)}(n, t)$ and $I_{\varepsilon,2}^{(1,1)}(n, t)$.

• **Case 1:** $1 - \cos \theta \gtrsim 1$. In this case, from (4.24), (4.23), and Lemma 2.3, we have

$$\begin{aligned} I_{\varepsilon,1}^{(1,1)}(n, t) &\lesssim_T \sum_{n=n_1+n_2} \frac{1}{\langle n \rangle^3 \langle n_1 \rangle^{1-2\alpha} \langle n_2 \rangle^{3-2\alpha}} \\ &\lesssim \langle n \rangle^{-4+2\alpha} \lesssim \langle n \rangle^{-2-2s_\alpha}, \end{aligned}$$

provided that $0 < \alpha < \frac{1}{2}$. This verifies (4.10).

• **Case 2:** $1 - \cos \theta \ll 1$. In this case, we have $0 \leq \theta \ll 1$, namely, n and n_2 point in almost opposite directions. In particular, we have $1 - \cos \theta \sim \theta^2 \ll 1$. By dyadically decomposing n_2 into $|n_2| \sim N_2$ for dyadic $N_2 \geq 1$, we have

$$I_{\varepsilon,2}^{(1,1)}(n, t) \lesssim_T \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \sum_{\substack{n=n_1+n_2 \\ \theta^2 \ll 1 \\ |n_2| \sim N_2}} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_1(\bar{n})|)}. \quad (4.25)$$

We see that for fixed $n \in \mathbb{Z}^2$, the range of possible n_2 with $|n_2| \sim N_2$ is constrained to an axially symmetric trapezoid \mathcal{R} whose height is $\sim N_2 \cos \theta \sim N_2$ and the top and bottom

widths $\sim N_2 \sin \theta \sim N_2 \theta$ with the axis of symmetry given by $-n$. See Figure 1. Hence, we have

$$\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ \theta^2 \ll 1 \\ |n_2| \sim N_2}} 1 \lesssim 1 + \text{vol}(\mathcal{R}) \sim 1 + N_2^2 \theta. \quad (4.26)$$

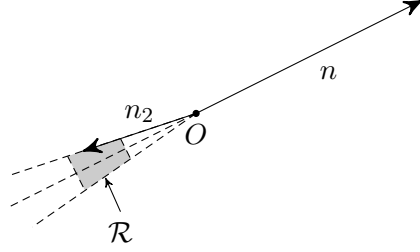


FIGURE 1. A typical configuration in Case 2

We now use the following seemingly crude bound:

$$1 + |\kappa_1(\bar{n})| \gtrsim |\kappa_1(\bar{n})|^{\frac{1}{2}}. \quad (4.27)$$

Then, from (4.23) and (4.27) with $|n_j| \sim N_j$, $j = 1, 2$, we have

$$\Theta_1 := \frac{1 + N_2^2 \theta}{1 + |\kappa_1(\bar{n})|} \lesssim N_1^{\frac{1}{2}} \frac{1 + N_2^2 \theta}{1 + \langle n \rangle^{\frac{1}{2}} N_2^{\frac{1}{2}} \theta} \lesssim \frac{N_1^{\frac{1}{2}} N_2^{\frac{3}{2}}}{\langle n \rangle^{\frac{1}{2}}}, \quad (4.28)$$

provided that $N_2^3 \gtrsim \langle n \rangle$. The bound (4.28) follows from separately considering the cases: $\langle n \rangle \lesssim N_2$ and $N_2 \ll \langle n \rangle \lesssim N_2^3$, using the condition $N_2^3 \gtrsim \langle n \rangle$. When $N_2^3 \ll \langle n \rangle$, the bound (4.28) also holds under an additional assumption $N_2^2 \theta \gtrsim 1$ (which also implies $\langle n \rangle^{\frac{1}{2}} N_2^{\frac{1}{2}} \theta \gg 1$). When $N_2^3 \ll \langle n \rangle$ and $N_2^2 \theta \ll 1$, we simply use the lower bound: $1 + |\kappa_1(\bar{n})| \gtrsim 1$.

Hence, from (4.25), (4.26), (4.27), and (4.28) with $N_1 \sim |n_1| \sim \max(|n|, |n_2|)$, we have

$$\begin{aligned} I_{\varepsilon, 2}^{(1,1)}(n, t) &\lesssim_T \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^2 N_1^{2(1-\alpha)} N_2^{2(1-\alpha)}} \Theta_1 \cdot (\mathbf{1}_{N_2^3 \gtrsim \langle n \rangle} + \mathbf{1}_{N_2^3 \ll \langle n \rangle} \cdot \mathbf{1}_{N_2^2 \theta \gtrsim 1}) \\ &\quad + \sum_{\substack{1 \ll N_2 \ll \langle n \rangle^{\frac{1}{3}} \\ \text{dyadic}}} \frac{1}{\langle n \rangle^2 N_1^{2(1-\alpha)} N_2^{2(1-\alpha)}} \cdot \mathbf{1}_{N_2^2 \theta \ll 1} \\ &\lesssim \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^{\frac{5}{2}} \max(\langle n \rangle, N_2)^{\frac{3}{2}-2\alpha} N_2^{\frac{1}{2}-2\alpha}} \\ &= \sum_{\substack{1 \leq N_2 < \langle n \rangle \\ \text{dyadic}}} \frac{1}{\langle n \rangle^{4-2\alpha} N_2^{\frac{1}{2}-2\alpha}} + \sum_{\substack{N_2 \geq \langle n \rangle \\ \text{dyadic}}} \frac{1}{\langle n \rangle^{\frac{5}{2}} N_2^{2-4\alpha}}. \end{aligned} \quad (4.29)$$

The first term on the right-hand side of (4.29) is bounded by

$$\begin{aligned} \sum_{\substack{1 \leq N_2 < \langle n \rangle \\ \text{dyadic}}} \frac{1}{\langle n \rangle^{4-2\alpha} N_2^{\frac{1}{2}-2\alpha}} &\lesssim \begin{cases} \langle n \rangle^{-4+2\alpha+}, & \text{if } \alpha \leq \frac{1}{4}, \\ \langle n \rangle^{-\frac{9}{2}+4\alpha}, & \text{if } \alpha > \frac{1}{4} \end{cases} \\ &\lesssim \langle n \rangle^{-2-2s_\alpha+}, \end{aligned}$$

where s_α is as in (1.13). As for the second term on the right-hand side of (4.29), we have

$$\sum_{\substack{N_2 \geq \langle n \rangle \\ \text{dyadic}}} \frac{1}{\langle n \rangle^{\frac{5}{2}} N_2^{2-4\alpha}} \lesssim \langle n \rangle^{-\frac{9}{2}+4\alpha} \lesssim \langle n \rangle^{-2-2s_\alpha}$$

for $\alpha < \frac{1}{2}$.

Lastly, when $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \pm 1, \pm 1)$, we can essentially reduce the analysis to Cases 1 and 2 above. See Case 3 in the proof of Proposition 1.6 in [30]. This completes the proof of Proposition 1.4.

Remark 4.1. As mentioned in Section 1, the extra smoothing for \mathbb{Y} on \mathbb{T}^2 is at most $\frac{1}{4}$, while $\frac{1}{2}$ -extra smoothing on \mathbb{T}^3 was shown in [30]. This difference comes from Case 2 in the proof of Proposition 1.4 above, where we applied (4.27). We point out that the bound (4.27) is used to cancel the powers of θ in (4.29). Furthermore, we can show that the estimate shown above is essentially sharp. More precisely, we have the following the lower bound of $I^{(1,1)}(n, t)$ in (4.17):

$$I^{(1,1)}(n, t) \gtrsim t^4 \langle n \rangle^{-2-2s_\alpha} \quad (4.30)$$

for $0 < \alpha < \frac{1}{2}$ and $\langle n \rangle^{-\frac{1}{2}} \ll t \ll 1$, where s_α is as in (1.13).

For simplicity, we drop the truncation $|n_1|, |n_2| \leq N$ in (4.17) with the understanding that a rigorous computation is to be done with the truncation $|n_1|, |n_2| \leq N$ in (4.17) and then by taking $N \rightarrow \infty$. Namely, we consider that $I^{(1,1)}(n, t)$ in (4.17) is written as follows:

$$\begin{aligned} I^{(1,1)}(n, t) &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \int_0^t \int_0^{t_1} \frac{\sin((t-t_1)\langle n \rangle)}{\langle n \rangle} \frac{\sin((t-t_2)\langle n \rangle)}{\langle n \rangle} \\ &\quad \times \frac{\cos((t_1-t_2)\langle n_1 \rangle)}{\langle n_1 \rangle^{2(1-\alpha)}} \frac{\cos((t_1-t_2)\langle n_2 \rangle)}{\langle n_2 \rangle^{2(1-\alpha)}} t_2^2 dt_2 dt_1. \end{aligned} \quad (4.31)$$

A direct calculation shows that

$$\begin{aligned} &\sin((t-t_1)\langle n \rangle) \sin((t-t_2)\langle n \rangle) \cos((t_1-t_2)\langle n_1 \rangle) \cos((t_1-t_2)\langle n_2 \rangle) \\ &= \frac{1}{4} \left(-\cos((2t-t_1-t_2)\langle n \rangle) + \cos((t_1-t_2)\langle n \rangle) \right) \\ &\quad \times \left(\cos((t_1-t_2)(\langle n_1 \rangle + \langle n_2 \rangle)) + \cos((t_1-t_2)(\langle n_1 \rangle - \langle n_2 \rangle)) \right) \\ &= \frac{1}{4} \left(-\cos((2t-t_1-t_2)\langle n \rangle) \cos((t_1-t_2)(\langle n_1 \rangle + \langle n_2 \rangle)) \right. \\ &\quad - \cos((2t-t_1-t_2)\langle n \rangle) \cos((t_1-t_2)(\langle n_1 \rangle - \langle n_2 \rangle)) \\ &\quad + \cos((t_1-t_2)\langle n \rangle) \cos((t_1-t_2)(\langle n_1 \rangle + \langle n_2 \rangle)) \\ &\quad \left. + \cos((t_1-t_2)\langle n \rangle) \cos((t_1-t_2)(\langle n_1 \rangle - \langle n_2 \rangle)) \right) \end{aligned}$$

$$=: \sum_{j=1}^4 A_{n,n_1,n_2}^{(j)}(t, t_1, t_2). \quad (4.32)$$

We denote the contribution to (4.31) from $A_{n,n_1,n_2}^{(j)}(t, t_1, t_2)$ by $I_j(n, t)$:

$$I_j(n, t) := \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \int_0^t \int_0^{t_1} A_{n,n_1,n_2}^{(j)}(t, t_1, t_2) t_2^2 dt_2 dt_1. \quad (4.33)$$

As we see below, the main contribution comes from $I_4(n, t)$.

First, we show that I_1 and I_2 satisfy

$$|I_1(n, t)| + |I_2(n, t)| \lesssim t^3 \langle n \rangle^{-\frac{5}{2}-2s_\alpha} \quad (4.34)$$

for $0 < \alpha < \frac{1}{2}$, $0 \leq t \leq 1$, and $n \in \mathbb{Z}^2$. In the following, we only estimate I_1 , since I_2 can be handled in an analogous manner. By applying a change of the variable $\tau_1 = \frac{t_1-t_2}{2}$, $\tau_2 = \frac{t_1+t_2}{2}$ to (4.33), we have

$$\begin{aligned} |I_1(n, t)| &\lesssim \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \\ &\times \left| \int_0^{\frac{t}{2}} \int_{\tau_1}^{t-\tau_1} \cos(2(t-\tau_2)\langle n \rangle) \cos(2\tau_1(\langle n_1 \rangle + \langle n_2 \rangle)) (\tau_1 - \tau_2)^2 d\tau_2 d\tau_1 \right|. \end{aligned} \quad (4.35)$$

By integration by parts, we can bound the inner integral by

$$\begin{aligned} &\left| \int_{\tau_1}^{t-\tau_1} \cos(2(t-\tau_2)\langle n \rangle) (\tau_1 - \tau_2)^2 d\tau_2 \right| \\ &= \left| -\frac{\sin(2\tau_1\langle n \rangle)}{2\langle n \rangle} (2\tau_1 - t)^2 \right. \\ &\quad \left. - \frac{1}{\langle n \rangle} \int_{\tau_1}^{t-\tau_1} \sin(2(t-\tau_2)\langle n \rangle) (\tau_1 - \tau_2) d\tau_2 \right| \\ &\lesssim \frac{t^2}{\langle n \rangle} \end{aligned} \quad (4.36)$$

for $0 \leq \tau_1 \leq \frac{t}{2} \leq \frac{1}{2}$. It follows from (4.35), (4.36), and Lemma 2.3 with $0 < \alpha < \frac{1}{2}$ that

$$\begin{aligned} |I_1(n, t)| &\lesssim \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{t^3}{\langle n \rangle^3 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \\ &\lesssim t^3 \langle n \rangle^{-5+4\alpha} \lesssim t^3 \langle n \rangle^{-\frac{5}{2}-2s_\alpha} \end{aligned}$$

for $0 \leq t \leq 1$ and $n \in \mathbb{Z}^2$. This proves (4.34).

Next, we prove the lower bound (4.30) on from $I^{(1,1)}(n, t)$. From (4.32) and the product formula, we have

$$\begin{aligned} A_{n,n_1,n_2}^{(3)}(t, t_1, t_2) &= \frac{1}{8} \left(\cos((t_1 - t_2)(\langle n \rangle + \langle n_1 \rangle + \langle n_2 \rangle)) \right. \\ &\quad \left. + \cos((t_1 - t_2)(\langle n \rangle - \langle n_1 \rangle - \langle n_2 \rangle)) \right), \\ A_{n,n_1,n_2}^{(4)}(t, t_1, t_2) &= \frac{1}{8} \left(\cos((t_1 - t_2)(\langle n \rangle + \langle n_1 \rangle - \langle n_2 \rangle)) \right. \\ &\quad \left. + \cos((t_1 - t_2)(\langle n \rangle - \langle n_1 \rangle + \langle n_2 \rangle)) \right). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_0^t \int_0^{t_1} \cos((t_1 - t_2)a) t_2^2 dt_2 dt_1 &= \int_0^t \frac{2}{a^2} \left(t_1 - \frac{\sin(t_1 a)}{a} \right) dt_1 \\ &= \frac{t^2}{a^2} + 2 \frac{\cos(ta) - 1}{a^4} \\ &= \frac{2}{a^4} \left(\cos(ta) - 1 + \frac{t^2 a^2}{2} \right) \geq 0 \end{aligned} \quad (4.37)$$

for any $a \in \mathbb{R} \setminus \{0\}$ and $t \geq 0$. When $a = 0$, then the left-hand side of (4.37) is obviously non-negative. Hence, from (4.33) and (4.37), we see that $I_3(n, t), I_4(n, t) \geq 0$. Hence, from (4.33) and (4.34), we obtain

$$\begin{aligned} I^{(1,1)}(n, t) &\geq I_4(n, t) - Ct^3 \langle n \rangle^{-\frac{5}{2}-2s_\alpha} \\ &\geq J(n, t) - Ct^3 \langle n \rangle^{-\frac{5}{2}-2s_\alpha} \end{aligned} \quad (4.38)$$

for $0 < t \ll 1$, where $J(n, t)$ is defined by

$$\begin{aligned} J(n, t) &:= \frac{1}{16\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \mathbf{1}_{n_2 \neq 0} \cdot \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \frac{1}{(\langle n \rangle - \langle n_1 \rangle + \langle n_2 \rangle)^4} \\ &\quad \times \left(\cos(t(\langle n \rangle - \langle n_1 \rangle + \langle n_2 \rangle)) - 1 + \frac{t^2(\langle n \rangle - \langle n_1 \rangle + \langle n_2 \rangle)^2}{2} \right) \\ &\quad + \frac{1}{768\pi^2} \frac{t^4}{\langle n \rangle^{4-2\alpha}}. \end{aligned} \quad (4.39)$$

Once we have

$$J(n, t) \gtrsim t^4 \langle n \rangle^{-2-2s_\alpha} \quad (4.40)$$

for $0 < t \ll 1$ and $n \in \mathbb{Z}^2$, (4.30) follows from (4.38) and (4.40):

$$I^{(1,1)}(n, t) \gtrsim t^3 \left(t - \langle n \rangle^{-\frac{1}{2}} \right) \langle n \rangle^{-2-2s_\alpha} \gtrsim t^4 \langle n \rangle^{-2-2s_\alpha}$$

for $\langle n \rangle^{-\frac{1}{2}} \ll t \ll 1$.

Hence, it remains to prove (4.40). First, consider the case $0 < \alpha \leq \frac{1}{4}$. In this case, from the second term on the right-hand side of (4.39), we have

$$J(n, t) \gtrsim t^4 \langle n \rangle^{-4+2\alpha} = t^4 \langle n \rangle^{-2-2(1-\alpha)} \quad (4.41)$$

for $0 < t \ll 1$. In view of (1.13), this proves (4.40) in this case.

Next, we consider the case $\frac{1}{4} < \alpha < \frac{1}{2}$. Given a dyadic number M , we set $n = (M, 0) \in \mathbb{Z}^2$ and

$$K_M = \left\{ (a, b) \in \mathbb{Z}^2 : 2M \leq a \leq 4M, |b| \leq M^{\frac{1}{2}} \right\}.$$

With $n_1 = (a, b) \in K_M$, we have

$$\begin{aligned} \langle n \rangle - M &= \frac{1}{\langle n \rangle + M} \lesssim M^{-1}, \\ \langle n_1 \rangle - a &= \frac{1 + b^2}{\langle n_1 \rangle + a} \lesssim 1, \\ \langle n - n_1 \rangle - (a - M) &= \frac{1 + b^2}{\langle n - n_1 \rangle + a - M} \lesssim 1. \end{aligned} \tag{4.42}$$

Then, it follows from (4.42) that

$$\langle n \rangle - \langle n_1 \rangle + \langle n - n_1 \rangle = M - a + (a - M) + O(1) \lesssim 1. \tag{4.43}$$

Hence, from (4.43) and the Taylor remainder theorem, we obtain

$$\begin{aligned} J(n, t) &\gtrsim \sum_{n_1 \in K_M} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^{2(1-\alpha)} \langle n - n_1 \rangle^{2(1-\alpha)}} \frac{1}{(\langle n \rangle - \langle n_1 \rangle + \langle n - n_1 \rangle)^4} \\ &\quad \times \left(\cos(t(\langle n \rangle - \langle n_1 \rangle + \langle n - n_1 \rangle)) - 1 + \frac{t^2(\langle n \rangle - \langle n_1 \rangle + \langle n - n_1 \rangle)^2}{2} \right) \\ &\sim t^4 M^{-\frac{9}{2} + 4\alpha} \sim t^4 \langle n \rangle^{-2 - 2(\frac{5}{4} - 2\alpha)} \end{aligned} \tag{4.44}$$

for $0 < t \ll 1$. In view of (1.13), this proves (4.40) when $\alpha > \frac{1}{4}$.

We also point out that the calculation above can easily be extended to the higher dimensional case. More precisely, the right-hand side of (4.41) is unchanged on \mathbb{T}^d since we did not perform any summation. By setting

$$K_M = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z}^{d-1} : 2M \leq a \leq 4M, |b| \leq M^{\frac{1}{2}} \right\}$$

and repeating the same computation on \mathbb{T}^d , the power on the right-hand side of (4.44) becomes $-\frac{11}{2} + \frac{d}{2} + 4\alpha$.

By writing

$$-4 + 2\alpha = -d - 2\left(2 - \frac{d}{2} - \alpha\right) \quad \text{and} \quad -\frac{11}{2} + \frac{d}{2} + 4\alpha = -d - 2\left(\frac{11}{4} - \frac{3d}{4} - 2\alpha\right),$$

this computation indicates that the regularity of \mathfrak{Y} on \mathbb{T}^d is at best

$$\min \left(2 - \frac{d}{2} - \alpha, \frac{11}{4} - \frac{3d}{4} - 2\alpha \right) = \begin{cases} 2 - \frac{d}{2} - \alpha, & \text{if } 0 < \alpha \leq \frac{3-d}{4}, \\ \frac{11}{4} - \frac{3d}{4} - 2\alpha, & \text{if } \alpha > \frac{3-d}{4}. \end{cases}$$

When $d = 3$ and $\alpha = 0$, this agrees with the $\frac{1}{2}$ -smoothing shown in [30].

4.3. Proof of Proposition 3.2. In this subsection, we present a proof of Proposition 3.2 on the resonant product $\mathfrak{Y}_N = \mathfrak{Y}_N \ominus \mathfrak{I}_N$. As in the previous subsection, we follow the argument in [30] but, as we see below, our argument turns out to be simpler than the proof of Proposition 1.8 in [30].

From (2.3) and (3.6), we have

$$\begin{aligned}
\widehat{\Psi}_N(n, t) &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_1+n_2| \sim |n_3| \\ n_1+n_2 \neq 0}} \int_0^t \frac{\sin((t-t')\langle n_1+n_2 \rangle)}{\langle n_1+n_2 \rangle} \widehat{\mathfrak{I}}_N(n_1, t') \widehat{\mathfrak{I}}_N(n_2, t') dt' \cdot \widehat{\mathfrak{I}}_N(n_3, t) \\
&\quad + \frac{1}{4\pi^2} \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ |n_1| \leq N}} \mathbf{1}_{|n| \sim 1} \int_0^t \sin(t-t') \cdot (|\widehat{\mathfrak{I}}(n_1, t')|^2 - \sigma_{n_1}(t')) dt' \cdot \widehat{\mathfrak{I}}_N(n, t) \\
&=: \widehat{\mathcal{R}}_1(n, t) + \widehat{\mathcal{R}}_2(n, t),
\end{aligned} \tag{4.45}$$

where the conditions $|n_1+n_2| \sim |n_3|$ in the first term and $|n| \sim 1$ in the second term signify the resonant product \ominus . From (4.3) and Lemma 2.6, we easily see that $\mathcal{R}_2 \in C(\mathbb{R}_+; C^\infty(\mathbb{T}^2))$ almost surely, provided that $\alpha < \frac{1}{2}$. Therefore, it suffices to show

$$\mathbb{E}[|\widehat{\mathcal{R}}_1(n, t)|^2] \lesssim_T \langle n \rangle^{-2-2(s_\alpha-\alpha)+} \tag{4.46}$$

for $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$, where s_α is as in (1.13). As in [30], decompose $\widehat{\mathcal{R}}_1$ as

$$\begin{aligned}
\widehat{\mathcal{R}}_1(n, t) &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_1+n_2| \sim |n_3| \\ (n_1+n_2)(n_2+n_3)(n_3+n_1) \neq 0}} \int_0^t \frac{\sin((t-t')\langle n_1+n_2 \rangle)}{\langle n_1+n_2 \rangle} \\
&\quad \times \widehat{\mathfrak{I}}_N(n_1, t') \widehat{\mathfrak{I}}_N(n_2, t') dt' \cdot \widehat{\mathfrak{I}}_N(n_3, t) \\
&\quad + \frac{1}{2\pi^2} \int_0^t \widehat{\mathfrak{I}}_N(n, t') \left[\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} \frac{\sin((t-t')\langle n+n_2 \rangle)}{\langle n+n_2 \rangle} \right. \\
&\quad \times \left(\widehat{\mathfrak{I}}(n_2, t') \widehat{\mathfrak{I}}(-n_2, t) - \sigma_{n_2}(t, t') \right) \Big] dt' \\
&\quad + \frac{1}{2\pi^2} \int_0^t \widehat{\mathfrak{I}}_N(n, t') \left[\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} \frac{\sin((t-t')\langle n+n_2 \rangle)}{\langle n+n_2 \rangle} \sigma_{n_2}(t, t') \right] dt' \\
&\quad - \frac{1}{4\pi^2} \cdot \mathbf{1}_{n \neq 0} \int_0^t \frac{\sin((t-t')\langle 2n \rangle)}{\langle 2n \rangle} (\widehat{\mathfrak{I}}_N(n, t'))^2 dt' \cdot \widehat{\mathfrak{I}}_N(-n, t) \\
&=: \widehat{\mathcal{R}}_{11}(n, t) + \widehat{\mathcal{R}}_{12}(n, t) + \widehat{\mathcal{R}}_{13}(n, t) + \widehat{\mathcal{R}}_{14}(n, t),
\end{aligned} \tag{4.47}$$

where \mathcal{R}_{12} and \mathcal{R}_{14} correspond to the “renormalized” contribution from $n_1+n_3=0$ or $n_2+n_3=0$ and the contribution from $n_1=n_2=n=-n_3$, respectively.

Proceeding as in [30] (and noting that $|n+n_2| \sim |n_2|$ implies $|n_2| \gtrsim |n|$), we can estimate $\widehat{\mathcal{R}}_{12}$ and $\widehat{\mathcal{R}}_{14}$ and show that they satisfy (4.46). As for $\widehat{\mathcal{R}}_{11}$, by applying Jensen’s inequality as in [30] (see also Section 10 in [33] and the discussion on Ψ in Section 4 of [40]) and then

(4.4), (4.10), and Lemma 2.3 (ii) (noting that $|m| \sim |n_3|$), we obtain

$$\begin{aligned} \mathbb{E}[|\widehat{\mathcal{R}}_{11}(n, t)|^2] &\lesssim \sum_{\substack{n=m+n_3 \\ |m| \sim |n_3|}} \mathbb{E}[|\widehat{\mathcal{V}}(m, t)|^2] \mathbb{E}[|\widehat{\mathcal{V}}(n_3, t)|^2] \\ &\lesssim^T \sum_{\substack{n=m+n_3 \\ |m| \sim |n_3|}} \frac{1}{\langle m \rangle^{2+2s_\alpha} \langle n_3 \rangle^{2-2\alpha}} \\ &\lesssim \langle n \rangle^{-2-2(s_\alpha-\alpha)+}, \end{aligned} \quad (4.48)$$

provided that $0 < \alpha < \frac{1}{2}$.

Lastly, we consider $\widehat{\mathcal{R}}_{13}$ in (4.47). Let $0 \leq t_2 \leq t_1 \leq T$. Then, from (4.47) with (4.1), we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathcal{R}}_{13}(n, t)|^2] &= \frac{1}{2\pi^4} \mathbf{1}_{|n| \leq N} \sum_{k_0, k_1, k_2 \in \{1, 2\}} \int_0^t \int_0^{t_1} \sigma_n^{(k_0)}(t_1, t_2) \\ &\quad \times \left[\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} \frac{\sin((t-t_1)\langle n+n_2 \rangle)}{\langle n+n_2 \rangle} \sigma_{n_2}^{(k_1)}(t, t_1) \right] \\ &\quad \times \left[\sum_{\substack{n'_2 \in \mathbb{Z}^2 \\ |n'_2| \sim |n+n'_2| \neq 0 \\ |n'_2| \leq N}} \frac{\sin((t-t_2)\langle n+n'_2 \rangle)}{\langle n+n'_2 \rangle} \sigma_{n'_2}^{(k_2)}(t, t_2) \right] dt_2 dt_1 \\ &=: \sum_{k_0, k_1, k_2 \in \{1, 2\}} \mathbf{I}^{(k_0, k_1, k_2)}(n, t), \end{aligned}$$

where $\sigma_n(t, t') = \sigma_n^{(1)}(t, t') + \sigma_n^{(2)}(t, t')$ as in (4.15) and (4.16). In the following, we only consider the contribution from $(k_0, k_1, k_2) = (1, 1, 1)$, since, in the other cases, the desired bound (4.46) trivially follows from Lemma 2.3 (ii) without using any oscillatory behavior.

By a direction computation with (4.15), we have

$$\begin{aligned} &\mathbf{I}^{(1,1,1)}(n, t) \\ &\sim \mathbf{1}_{|n| \leq N} \sum_{\substack{\varepsilon_j \in \{-1, 1\} \\ j=1, \dots, 5}} \sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} \sum_{\substack{n'_2 \in \mathbb{Z}^2 \\ |n'_2| \sim |n+n'_2| \neq 0 \\ |n'_2| \leq N}} \frac{\varepsilon_1 \varepsilon_2 e^{it(\varepsilon_1 \langle n+n_2 \rangle + \varepsilon_2 \langle n+n'_2 \rangle + \varepsilon_3 \langle n_2 \rangle + \varepsilon_4 \langle n'_2 \rangle)}}{\langle n \rangle^{2(1-\alpha)} \langle n+n_2 \rangle \langle n_2 \rangle^{2(1-\alpha)} \langle n+n'_2 \rangle \langle n'_2 \rangle^{2(1-\alpha)}} \\ &\quad \times \int_0^t t_1 e^{-it_1 \kappa_3(\bar{n})} \int_0^{t_1} t_2^2 e^{-it_2 \kappa_4(\bar{n}')} dt_2 dt_1, \end{aligned} \quad (4.49)$$

where $\kappa_3(\bar{n})$ and $\kappa_4(\bar{n}')$ are defined by

$$\begin{aligned} \kappa_3(\bar{n}) &= \varepsilon_1 \langle n+n_2 \rangle + \varepsilon_3 \langle n_2 \rangle - \varepsilon_5 \langle n \rangle, \\ \kappa_4(\bar{n}') &= \varepsilon_2 \langle n+n'_2 \rangle + \varepsilon_4 \langle n'_2 \rangle + \varepsilon_5 \langle n \rangle. \end{aligned}$$

Note that for $\alpha < \frac{1}{2}$, the sums over n_2 and n'_2 in (4.49) are absolutely convergent. This makes our analysis simpler than the proof of Proposition 1.8 in [30], where the corresponding

sums in n_2 and n'_2 were not absolutely convergent and hence, it was crucial to exploit the oscillatory nature of the problem and also apply some symmetrization argument.

By first integrating (4.49) in t_1 when $|\kappa_3(\bar{n})| \geq 1$ and simply bounding the integral in (4.49) by $C(T)$ when $|\kappa_3(\bar{n})| < 1$, and then applying Lemma 2.3 (ii), we have

$$\begin{aligned}
|I^{(1,1,1)}(n, t)| &\lesssim_T \frac{1}{\langle n \rangle^{2(1-\alpha)}} \sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n+n_2| \sim |n_2|}} \frac{1}{\langle n+n_2 \rangle \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_3(\bar{n})|)} \\
&\quad \times \sum_{\substack{n'_2 \in \mathbb{Z}^2 \\ |n+n'_2| \sim |n'_2|}} \frac{1}{\langle n+n'_2 \rangle \langle n'_2 \rangle^{2(1-\alpha)}} \\
&\lesssim \frac{1}{\langle n \rangle^{3-4\alpha}} \sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n+n_2| \sim |n_2|}} \frac{1}{\langle n+n_2 \rangle \langle n_2 \rangle^{2(1-\alpha)} (1 + |\kappa_3(\bar{n})|)} \tag{4.50}
\end{aligned}$$

for $\alpha < \frac{1}{2}$. In the following, we only consider the case $(\varepsilon_1, \varepsilon_3, \varepsilon_5) = (\pm 1, \mp 1, \pm 1)$, since the other cases are handled in an analogous manner. See also the proof of Proposition 1.4.

In this case, by repeating the argument in Case 2 of the proof of Proposition 1.4 (in particular, (4.23) with $(n, n+n_2, -n_2)$ replacing (n, n_1, n_2)), we have

$$|\kappa_3(\bar{n})| = |\langle n+n_2 \rangle - \langle n_2 \rangle - \langle n \rangle| \gtrsim \frac{1 + |n||n_2|(1 - \cos \theta)}{\langle n+n_2 \rangle}, \tag{4.51}$$

where $\theta = \angle(n, n_2)$. Then, as in (4.28), it follows from (4.51) and $1 + |\kappa_3(\bar{n})| \gtrsim |\kappa_3(\bar{n})|^{\frac{1}{2}}$ with $|n_2| \sim N_2$ that

$$\Theta_3 := \frac{1 + N_2^2 \theta}{1 + |\kappa_3(\bar{n})|} \lesssim \langle n+n_2 \rangle^{\frac{1}{2}} \frac{1 + N_2^2 \theta}{1 + \langle n \rangle^{\frac{1}{2}} N_2^{\frac{1}{2}} \theta} \lesssim \frac{N_2^2}{\langle n \rangle^{\frac{1}{2}}}, \tag{4.52}$$

since $N_2^3 \gtrsim \langle n \rangle$ under the condition $|n+n_2| \sim |n_2|$.

When $1 - \cos \theta \gtrsim 1$, by summing over n_2 in (4.50) with (4.51) and Lemma 2.3, we obtain

$$|I^{(1,1,1)}(n, t)| \lesssim_T \langle n \rangle^{-5+6\alpha} \leq \langle n \rangle^{-2-2(s_\alpha-\alpha)}$$

for $\alpha < \frac{1}{2}$.

Next, consider the case $1 - \cos \theta \sim \theta^2 \ll 1$. We see that for fixed $n \in \mathbb{Z}^2$, the range of possible n_2 with $|n_2| \sim N_2$, dyadic $N_2 \geq 1$, is constrained to a trapezoid whose height is $\sim N_2 |\cos \theta| \sim N_2$ and the width $\sim N_2 \sin \theta \lesssim N_2 \theta$. Then, from (4.50) with a dyadic decomposition as in (4.25) and (4.26), and (4.52), we have

$$\begin{aligned}
|I^{(1,1,1)}(n, t)| &\lesssim_T \frac{1}{\langle n \rangle^{3-4\alpha}} \sum_{\substack{N_2 \gtrsim \langle n \rangle \\ \text{dyadic}}} \frac{1}{N_2^{3-2\alpha}} \Theta_3 \\
&\lesssim \frac{1}{\langle n \rangle^{\frac{7}{2}-4\alpha}} \sum_{\substack{N_2 \gtrsim \langle n \rangle \\ \text{dyadic}}} \frac{1}{N_2^{1-2\alpha}} \\
&\lesssim \langle n \rangle^{-\frac{9}{2}+6\alpha} \lesssim \langle n \rangle^{-2-2(s_\alpha-\alpha)}
\end{aligned}$$

for $\alpha < \frac{1}{2}$. We therefore obtain (4.46).

Since the summations above are absolutely convergent, a slight modification of the argument yields the time difference estimate (2.11) and the estimates (2.12) and (2.13) for proving convergence of \mathfrak{Y}_N to \mathfrak{Y} by Lemma 2.6. This completes the proof of Proposition 3.2.

Remark 4.2. As pointed above, the sums in (4.49) is absolutely convergent for $\alpha < \frac{1}{2}$. Therefore, even without exploiting multilinear dispersion, we can make sense of \mathfrak{Y} . In the following, by a crude estimate, we show

$$\mathfrak{Y} \in C([0, T]; W^{1-3\alpha-, \infty}(\mathbb{T}^2)) \quad (4.53)$$

almost surely.

Note that $\widehat{\mathcal{R}}_{12}$ and $\widehat{\mathcal{R}}_{14}$ satisfy (4.46) without making use of any dispersion. Thus, we only need to consider $\widehat{\mathcal{R}}_{11}$ and $\widehat{\mathcal{R}}_{13}$. Let $I(n, t)$ be as in (4.12). Then, by applying Lemma 2.3 to (4.14) with (4.1), we have

$$|I(n, t)| \lesssim_T \frac{1}{\langle n \rangle^2} \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \lesssim \langle n \rangle^{-4+4\alpha}$$

for $0 < \alpha < \frac{1}{2}$. Together with (4.13), this implies

$$\mathbb{E}[|\widehat{\mathcal{Y}}(n, t)|^2] \lesssim_T \langle n \rangle^{-4+4\alpha} \quad (4.54)$$

even when we do not exploit multilinear dispersion. Then, using (4.4) and (4.54), we can replace (4.48) by

$$\begin{aligned} \mathbb{E}[|\widehat{\mathcal{R}}_{11}(n, t)|^2] &\lesssim \sum_{\substack{n=m+n_3 \\ |m| \sim |n_3|}} \mathbb{E}[|\widehat{\mathcal{Y}}(m, t)|^2] \mathbb{E}[|\widehat{\mathcal{I}}(n_3, t)|^2] \\ &\lesssim_T \langle n \rangle^{-4+6\alpha} = \langle n \rangle^{-2-2(1-3\alpha)}. \end{aligned} \quad (4.55)$$

By ignoring all the oscillatory factors in (4.49), we obtain

$$\begin{aligned} \mathbb{E}[|\widehat{\mathcal{R}}_{13}(n, t)|^2] &\lesssim_T \frac{1}{\langle n \rangle^{2(1-\alpha)}} \sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n+n_2| \sim |n_2|}} \frac{1}{\langle n+n_2 \rangle \langle n_2 \rangle^{2(1-\alpha)}} \sum_{\substack{n'_2 \in \mathbb{Z}^2 \\ |n+n'_2| \sim |n'_2|}} \frac{1}{\langle n+n'_2 \rangle \langle n'_2 \rangle^{2(1-\alpha)}} \\ &\lesssim \langle n \rangle^{-4+6\alpha}. \end{aligned} \quad (4.56)$$

Therefore, (4.53) follows from Lemma 2.6 with (4.55), (4.56), and the trivial bounds for $\widehat{\mathcal{R}}_{12}$ and $\widehat{\mathcal{R}}_{14}$.

4.4. Divergence of the stochastic terms. In this subsection, we present the proof of Proposition 1.6. By (3.6) and (3.4) with (4.2), for $n \in \mathbb{Z}^2$ and $t > 0$, we can write

$$\widehat{\mathcal{Y}}_N(n, t) = \frac{1}{2\pi} \sum_{\substack{k \in \mathbb{Z}^2 \\ k \preceq n-k \\ |k|, |n-k| \leq N}} X_k(n, t), \quad (4.57)$$

where \preceq denotes the lexicographic ordering of \mathbb{Z}^2 and

$$X_k(n, t) := (2 - \mathbf{1}_{n=2k}) \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle} \left(\widehat{\mathcal{I}}(k, t') \widehat{\mathcal{I}}(n-k, t') - \mathbf{1}_{n=0} \cdot \sigma_k(t', t') \right) dt'. \quad (4.58)$$

Note that $X_k(n, t)$'s are independent. We show that the sum in (4.57) diverges almost surely. We only consider the case $|k| \sim |n-k| \gg |n|$. Otherwise, we have either $|k| \sim |n| \gtrsim |n-k|$

or $|n - k| \sim |n| \gtrsim |k|$. In either case, for fixed $n \in \mathbb{Z}^2$, the sum in k is a finite sum and hence is almost surely convergent. This allows us to focus on the case $|k| \sim |n - k| \gg |n|$. In particular, we assume $k \neq \frac{n}{2}$.

As in (4.12), we have

$$\begin{aligned} \mathbb{E}[X_k(n, t)] &= 0, \\ \mathbb{E}[|X_k(n, t)|^2] &= 8 \int_0^t \frac{\sin((t - t_1)\langle n \rangle)}{\langle n \rangle} \int_0^{t_1} \frac{\sin((t - t_2)\langle n \rangle)}{\langle n \rangle} \\ &\quad \times \sigma_k(t_1, t_2) \sigma_{n-k}(t_1, t_2) dt_2 dt_1. \end{aligned} \quad (4.59)$$

When $n = 0$ (which implies $k \neq 0$ under the assumption $k \neq \frac{n}{2}$), we used (4.7). From (4.1) and $|k| \sim |n - k|$, we have

$$\begin{aligned} \sigma_k(t_1, t_2) \sigma_{n-k}(t_1, t_2) &= \frac{\cos((t_1 - t_2)(\langle k \rangle - \langle n - k \rangle))}{8\langle k \rangle^{2(1-\alpha)} \langle n - k \rangle^{2(1-\alpha)}} t_2^2 \\ &\quad + \frac{\cos((t_1 - t_2)(\langle k \rangle + \langle n - k \rangle))}{8\langle k \rangle^{2(1-\alpha)} \langle n - k \rangle^{2(1-\alpha)}} t_2^2 \\ &\quad + O(\langle t_2 \rangle \langle k \rangle^{-5+4\alpha}). \end{aligned} \quad (4.60)$$

The contribution to (4.59) from the first term on the right-hand side of (4.60) is worst. Indeed, we can use the dispersion to estimate the contribution to (4.59) from the second term on the right-hand side of (4.60). Namely, by integrating in t_2 and using $|k| \sim |n - k| \gg |n|$, we have

$$\begin{aligned} &\left| \int_0^t \frac{\sin((t - t_1)\langle n \rangle)}{\langle n \rangle} \int_0^{t_1} \frac{\sin((t - t_2)\langle n \rangle)}{\langle n \rangle} \frac{\cos((t_1 - t_2)(\langle k \rangle + \langle n - k \rangle))}{\langle k \rangle^{2(1-\alpha)} \langle n - k \rangle^{2(1-\alpha)}} t_2^2 dt_2 dt_1 \right| \\ &\lesssim \frac{1}{\langle n \rangle^2 \langle k \rangle^{2(1-\alpha)} \langle n - k \rangle^{2(1-\alpha)}} \\ &\quad \times \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} \int_0^t \left| \int_0^{t_1} e^{-it_2(\varepsilon_1 \langle n \rangle + \varepsilon_2(\langle k \rangle + \langle n - k \rangle))} t_2^2 dt_2 \right| dt_1 \\ &\lesssim_t \frac{1}{\langle n \rangle^2 \langle k \rangle^{5-4\alpha}}. \end{aligned} \quad (4.61)$$

Now, let us estimate the contribution to (4.59) from the first term on the right-hand side of (4.60). Given $n \in \mathbb{Z}^2$, we choose small $t > 0$ such that $t\langle n \rangle \ll 1$, which implies

$$\frac{\sin(t\langle n \rangle)}{\langle n \rangle} \gtrsim t \quad \text{and} \quad \cos(t\langle n \rangle) \gtrsim 1. \quad (4.62)$$

Noting that

$$|\langle k \rangle - \langle n - k \rangle| = \frac{||k|^2 - |n - k|^2|}{\langle k \rangle + \langle n - k \rangle} \lesssim |n|,$$

it follows from (4.62) that

$$\begin{aligned}
& \int_0^t \frac{\sin((t-t_1)\langle n \rangle)}{\langle n \rangle} \int_0^{t_1} \frac{\sin((t-t_2)\langle n \rangle)}{\langle n \rangle} \frac{\cos((t_1-t_2)(\langle k \rangle - \langle n-k \rangle))}{\langle k \rangle^{2(1-\alpha)} \langle n-k \rangle^{2(1-\alpha)}} t_2^2 dt_2 dt_1 \\
& \gtrsim \frac{1}{\langle k \rangle^{4(1-\alpha)}} \int_0^t (t-t_1) \int_0^{t_1} (t-t_2) t_2^2 dt_2 dt_1 \\
& \gtrsim \frac{t^6}{\langle k \rangle^{4(1-\alpha)}}.
\end{aligned} \tag{4.63}$$

By (4.59), (4.60), (4.61), and (4.63), we obtain

$$\mathbb{E}[|X_k(n, t)|^2] \gtrsim \frac{t^6}{\langle k \rangle^{4(1-\alpha)}} - \frac{1}{\langle n \rangle^2 \langle k \rangle^{5-4\alpha}} \gtrsim \frac{t^6}{\langle k \rangle^{4(1-\alpha)}} \tag{4.64}$$

for $|k| \gg t^{-6} \langle n \rangle^{-2}$ and $t \langle n \rangle \ll 1$. This implies that

$$\begin{aligned}
& \sum_{\substack{k \in \mathbb{Z}^2 \\ k \lesssim n-k \\ |n| \leq |k| \leq N}} \mathbb{E}[|X_k(n, t)|^2] \gtrsim_{t,n} \sum_{\substack{k \in \mathbb{Z}^2 \\ t^{-6} \langle n \rangle^{-2} \ll |k| \leq N}} \frac{1}{\langle k \rangle^{4(1-\alpha)}} \gtrsim_{t,n} \begin{cases} \log N, & \text{if } \alpha = \frac{1}{2} \\ N^{-2+4\alpha}, & \text{if } \alpha > \frac{1}{2} \end{cases} \\
& \longrightarrow \infty
\end{aligned}$$

as $N \rightarrow \infty$. Hence, Kolmogorov's three-series theorem ([23, Theorem 2.5.8]) yields that $P(|\lim_{N \rightarrow \infty} \hat{\Upsilon}_N(n, t)| < \infty) < 1$. Moreover, recalling the independence of $\{X_k(n, t)\}_{k \in \mathbb{Z}^2, k \lesssim n-k}$, it follows from Kolmogorov's zero-one law ([23, Theorem 2.5.3]) that

$$P\left(\lim_{N \rightarrow \infty} |\hat{\Upsilon}_N(n, t)| < \infty\right) = 0.$$

In particular, we obtain that $\{\Upsilon_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ almost surely for any $T > 0$.

Remark 4.3. (i) From (4.12), (4.13), and (4.59) we have

$$\mathbb{E}[|\hat{\Upsilon}_N(n, t)|^2] = \frac{1}{4\pi^2} \sum_{\substack{k \in \mathbb{Z}^2 \\ k \lesssim n-k \\ |k|, |n-k| \leq N}} \mathbb{E}[|X_k(n, t)|^2] + \mathbf{1}_{n \in 2\mathbb{Z}^2 \setminus \{0\}} \cdot O(\langle n \rangle^{-6+4\alpha}).$$

This shows that Proposition 1.6 is a consequence of $\lim_{N \rightarrow \infty} \mathbb{E}[|\hat{\Upsilon}_N(n, t)|^2] = \infty$.

(ii) Note that the calculations in (4.61), (4.62), and (4.63) are independent of dimensions. In particular, the lower bound (4.64) is also valid on \mathbb{T}^d . From this observation, we conclude that $\{\Upsilon_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^d))$ almost surely if $\alpha \geq 1 - \frac{d}{4}$. Note that for $d \geq 5$, we need to apply smoothing (i.e. $\alpha < 0$) in order to construct the second order process Υ as a limit of $\{\Upsilon_N\}_{N \in \mathbb{N}}$.

Since the critical value given by the probabilistic scaling is $\alpha_* = \min\left(\frac{5-d}{4}, \frac{5-d}{2}\right)$, we see that the existing solution theory such as the Da Prato-Debussche trick or its higher order variants breaks down at $\alpha = 1 - \frac{d}{4}$ before reaching the critical value α_* in dimensions $d = 1, \dots, 5$.

5. STOCHASTIC NONLINEAR HEAT EQUATION WITH ROUGH NOISE

In this section, we consider SNLH (1.2). In Subsection 5.1, we first state the regularity properties of the relevant stochastic terms and present a proof of Theorem 1.10 by reformulating the problem in terms of the residual term $v = u - \mathfrak{I} + \mathfrak{Y}$. We then proceed with the construction of the stochastic terms in the remaining part of this section. This includes the divergence of \mathfrak{V} (and \mathfrak{Y} , respectively) for $\alpha \geq \frac{1}{2}$ (and $\alpha \geq 1$, respectively) stated in Proposition 1.9.

5.1. Reformulation of SNLH. Let $\alpha > 0$. We define the truncated stochastic convolution $\mathfrak{I}_N = \mathcal{I}(\langle \nabla \rangle^\alpha \pi_N \xi)$ by

$$\mathfrak{I}_N := \int_{-\infty}^t P(t-t') \langle \nabla \rangle^\alpha \pi_N dW(t') = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} e_n \int_{-\infty}^t e^{-(t-t')\langle n \rangle^2} \langle n \rangle^\alpha d\beta_n(t') \quad (5.1)$$

for $t \geq 0$, where π_N , $P(t)$, e_n , and $W(t)$ are as in (1.7), (2.8), (2.1), and (3.1), respectively. We denote the limit of \mathfrak{I}_N by \mathfrak{I} :

$$\mathfrak{I} = \lim_{N \rightarrow \infty} \mathfrak{I}_N = \int_{-\infty}^t P(t-t') \langle \nabla \rangle^\alpha dW(t'). \quad (5.2)$$

We then define the truncated Wick power \mathfrak{V}_N by

$$\mathfrak{V}_N := (\mathfrak{I}_N)^2 - \kappa_N. \quad (5.3)$$

where κ_N is defined by

$$\begin{aligned} \kappa_N &:= \mathbb{E}[(\mathfrak{I}_N(x, t))^2] = \frac{1}{4\pi^2} \sum_{|n| \leq N} \int_{-\infty}^t \left(e^{-(t-t')\langle n \rangle^2} \langle n \rangle^\alpha \right)^2 dt' \\ &= \frac{1}{8\pi^2} \sum_{|n| \leq N} \frac{1}{\langle n \rangle^{2(1-\alpha)}} \sim N^{2\alpha}. \end{aligned} \quad (5.4)$$

Then, by proceeding as in the proof of Lemma 3.1 (i), we have the following regularity and convergence property of \mathfrak{I}_N . Since the argument is standard, we omit details.

Lemma 5.1. *Let $T > 0$. Given $\alpha \in \mathbb{R}$ and $s < -\alpha$, $\{\mathfrak{I}_N\}_{N \in \mathbb{N}}$ defined in (5.1) is a Cauchy sequence in $C([0, T]; \mathcal{C}^s(\mathbb{T}^2))$ almost surely. In particular, denoting the limit by \mathfrak{I} , we have*

$$\mathfrak{I} \in C([0, T]; \mathcal{C}^{-\alpha-\varepsilon}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

We now define the second order stochastic term:

$$\mathfrak{Y}_N := \mathcal{I}(\mathfrak{V}_N).$$

Then, a slight modification of the proof of Proposition 1.9 (ii) presented below shows that \mathfrak{Y}_N converges to

$$\mathfrak{Y}(t) := \mathcal{I}(\mathfrak{V})(t) = \int_{-\infty}^t P(t-t') \mathfrak{V}(t') dt'$$

in $C([0, T]; \mathcal{C}^{2-2\alpha-}(\mathbb{T}^2))$ almost surely, provided that $0 < \alpha < 1$.

From the regularities $2 - 2\alpha -$ and $-\alpha -$ of \mathcal{Y} and \mathfrak{I} , there is an issue in making sense of the resonant product $\mathcal{Y} \odot \mathfrak{I}$ in the deterministic manner when $\alpha \geq \frac{2}{3}$. For the range $\frac{2}{3} \leq \alpha < 1$, we instead use stochastic analysis to define the resonant product “ $\mathcal{Y} \odot \mathfrak{I}$ ” as a suitable limit of $\mathcal{Y}_N := \mathcal{Y}_N \odot \mathfrak{I}_N$.

Lemma 5.2. *Let $0 < \alpha < 1$. Given any $T > 0$, $\{\mathcal{Y}_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; \mathcal{C}^{2-3\alpha-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$, almost surely. In particular, denoting the limit by \mathcal{Y} , we have*

$$\mathcal{Y} \in C([0, T]; \mathcal{C}^{2-3\alpha-\varepsilon}(\mathbb{T}^2))$$

almost surely.

In the following, we only consider the range $\frac{2}{3} \leq \alpha < 1$ since the case $0 < \alpha < \frac{2}{3}$ can be handled by the standard Da Prato-Debussche trick as mentioned in Section 1. As in Subsection 3.1, we proceed with the second order expansion (1.3). Then, after a proper renormalization, the residual term $v = u - \mathfrak{I} + \mathcal{Y}$ satisfies the following equation:

$$\begin{cases} \partial_t v + (1 - \Delta)v = -v^2 - 2v(\mathfrak{I} - \mathcal{Y}) - \mathcal{Y}^2 + 2(\mathcal{Y} \odot \mathfrak{I} + \mathcal{Y} + \mathcal{Y} \odot \mathfrak{I}) \\ v|_{t=0} = v_0, \end{cases} \quad (5.5)$$

where $v_0 = v_0^\omega$ is given by

$$v_0 = u_0 - \mathfrak{I}(0) + \mathcal{Y}(0). \quad (5.6)$$

Given $s < \sigma$ and $T > 0$, define $X(T) \subset C([0, T]; \mathcal{C}^s(\mathbb{T}^2)) \cap C((0, T]; \mathcal{C}^\sigma(\mathbb{T}^2))$ by the norm:

$$\|v\|_{X(T)} = \|v\|_{C_T \mathcal{C}_x^s} + \|v\|_{Y(T)},$$

where the $Y(T)$ -norm is given by

$$\|v\|_{Y(T)} = \sup_{0 < t \leq T} t^{\frac{\sigma-s}{2}} \|v(t)\|_{\mathcal{C}^\sigma}.$$

We point out that the $Y(T)$ -norm is needed to handle rough initial data in $\mathcal{C}^s(T^2)$, which does not belong to $\mathcal{C}^\sigma(\mathbb{T}^2)$. The use of this type of norm, allowing a blowup at time $t = 0$, is standard in the study of the parabolic equations. See, for example, [6, 39]. We then have the following local well-posedness of the perturbed SNLH (5.5).

Theorem 5.3. *Let $0 < \alpha < 1$ and $s > -\alpha - \varepsilon$ for sufficiently small $\varepsilon > 0$. Then, the Cauchy problem (5.5) is locally well-posed in $\mathcal{C}^s(\mathbb{T}^2)$. More precisely, given any $u_0 \in \mathcal{C}^s(\mathbb{T}^2)$, there exist an almost surely positive stopping time $T = T(\omega)$ and a unique solution v to (5.5) in the class:*

$$X(T) \subset C([0, T]; \mathcal{C}^s(\mathbb{T}^2)) \cap C((0, T]; \mathcal{C}^\sigma(\mathbb{T}^2)),$$

where $-s < \sigma < s + 2$. Furthermore, the solution v depends continuously on the enhanced data set:

$$\Sigma = (u_0, \mathfrak{I}, \mathcal{Y}, \mathcal{Y})$$

almost surely belonging to the class:

$$\begin{aligned} \mathcal{Z}_T^{s, \varepsilon} &= \mathcal{C}^s(\mathbb{T}^2) \times C([0, T]; \mathcal{C}^{-\alpha-\varepsilon}(\mathbb{T}^2)) \\ &\quad \times C([0, T]; \mathcal{C}^{2-2\alpha-\varepsilon}(\mathbb{T}^2)) \times C([0, T]; \mathcal{C}^{2-3\alpha-\varepsilon, \infty}(\mathbb{T}^2)). \end{aligned}$$

Once we prove Theorem 5.3, Theorem 1.10 follows from the same lines as in the proof of Theorem 1.3 and thus we omit details.

Proof of Theorem 5.3. Let $0 < T \leq 1$ and fix $\varepsilon > 0$ sufficiently small. Define a map Γ on $X(T)$ by

$$\Gamma(v)(t) = P(t)v_0 - \int_0^t P(t-t') [v^2 + 2v(\mathfrak{I} - \mathfrak{Y}) + \mathfrak{Y}^2 - 2\mathfrak{Y}\mathfrak{Z}](t') dt',$$

where v_0 is as in (5.6) and \mathfrak{Y} is as in (3.16). From Lemma 5.2, Proposition 1.9, and Lemma 5.1 with Lemma 2.1, we see that

$$\mathfrak{Y} \in C([0, T]; \mathcal{C}^{-\alpha-\varepsilon, \infty}(\mathbb{T}^2)) \quad (5.7)$$

almost surely.

For simplicity, we only consider the case $s = -\alpha - \varepsilon$. From Lemmas 2.5 and 2.1 along with Lemma 5.1, Proposition 1.9, and (5.7), we have

$$\begin{aligned} \|\Gamma(v)\|_{L_T^\infty \mathcal{C}_x^s} &\lesssim \|v_0\|_{\mathcal{C}^s} + \left\| \int_0^t \left(\|v^2\|_{\mathcal{C}^s} + \|v(\mathfrak{I} - \mathfrak{Y})\|_{\mathcal{C}^s} \right) (t') dt' \right\|_{L_T^\infty} \\ &\quad + T \left(\|\mathfrak{Y}^2\|_{L_T^\infty \mathcal{C}_x^s} + \|\mathfrak{Y}\mathfrak{Z}\|_{L_T^\infty \mathcal{C}_x^s} \right) \\ &\lesssim \|u_0\|_{\mathcal{C}^s} + \|\mathfrak{I}(0)\|_{\mathcal{C}^s} + \|\mathfrak{Y}(0)\|_{\mathcal{C}^s} \\ &\quad + \int_0^T (t')^{-\frac{\sigma-s}{2}} dt' \cdot (\|v\|_{L_T^\infty \mathcal{C}_x^s} + \|\mathfrak{I} - \mathfrak{Y}\|_{L_T^\infty \mathcal{C}_x^s}) \|v\|_{Y(T)} \\ &\quad + T \left(\|\mathfrak{Y}\|_{L_T^\infty \mathcal{C}_x^{2-2\alpha-}}^2 + \|\mathfrak{Y}\mathfrak{Z}\|_{L_T^\infty \mathcal{C}_x^s} \right) \\ &\leq \|u_0\|_{\mathcal{C}^s} + T^\theta \left(\|v\|_{X(T)}^2 + C_\omega \|v\|_{X(T)} \right) + C_\omega \end{aligned} \quad (5.8)$$

for some almost surely finite constant $C_\omega > 0$ and $\theta > 0$, provided that $\alpha < 1$ and $-s < \sigma < s + 2$.

Next, we estimate the $Y(T)$ -norm of $\Gamma(v)$. Under the condition $\sigma < s + 2$, a change of variable yields

$$t^{\frac{\sigma-s}{2}} \int_0^t (t-t')^{-\frac{\sigma-s}{2}} (t')^{-\frac{\sigma-s}{2}} dt' = t^{1-\frac{\sigma-s}{2}} B\left(1 - \frac{\sigma-s}{2}, 1 - \frac{\sigma-s}{2}\right) \lesssim T^{1-\frac{\sigma-s}{2}} \quad (5.9)$$

for $0 < t \leq T$, provided that $\sigma < s + 2$. Here, $B(x, y) = \int_0^1 (1-\tau)^{x-1} \tau^{y-1} d\tau$ denotes the beta function.

Let $\mathcal{N}(v) = v^2 + 2v(\mathfrak{I} - \mathfrak{Y}) + \mathfrak{Y}^2 - 2\mathfrak{Y}\mathfrak{Z}$. Then, for $0 < t \leq T$, it follows from Lemmas 2.5 and 2.1 and (5.9) along with Lemma 5.1, Proposition 1.9, and (5.7) that

$$\begin{aligned} t^{\frac{\sigma-s}{2}} \|\Gamma(v)(t)\|_{\mathcal{C}_x^s} &\lesssim \|v_0\|_{\mathcal{C}^s} + t^{\frac{\sigma-s}{2}} \int_0^t (t-t')^{-\frac{\sigma-s}{2}} \|\mathcal{N}(v)(t')\|_{\mathcal{C}^s} dt' \\ &\lesssim \|u_0\|_{\mathcal{C}^s} + t^{\frac{\sigma-s}{2}} \int_0^t (t-t')^{-\frac{\sigma-s}{2}} (t')^{-\frac{\sigma-s}{2}} dt' \\ &\quad \times (\|v\|_{L_T^\infty \mathcal{C}_x^s} + \|\mathfrak{I} - \mathfrak{Y}\|_{L_T^\infty \mathcal{C}_x^s}) \|v\|_{Y(T)} + C_\omega \\ &\leq \|u_0\|_{\mathcal{C}^s} + T^\theta \left(\|v\|_{X(T)}^2 + C_\omega \|v\|_{X(T)} \right) + C_\omega, \end{aligned} \quad (5.10)$$

provided that $\alpha < 1$ and $-s < \sigma < s + 2$.

Taking a supremum of the left-hand side of (5.10) over $0 < t \leq T$, it follows from (5.8) and (5.10) that

$$\|\Gamma(v)\|_{X(T)} \leq \|u_0\|_{\mathcal{C}^s} + T^\theta \left(\|v\|_{X(T)}^2 + C_\omega \|v\|_{X(T)} \right) + C_\omega. \quad (5.11)$$

By a similar computation, we also obtain a difference estimate:

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{X(T)} \lesssim T^\theta \left(\|v_1\|_{X(T)} + \|v_2\|_{X(T)} + C_\omega \right) \|v_1 - v_2\|_{X(T)}. \quad (5.12)$$

Therefore, we conclude from (5.11) and (5.12) that a standard contraction argument yields local well-posedness of (5.5). Moreover, an analogous computation shows that the solution $v \in X(T)$ depends continuously on the enhanced data set $\Sigma = (u_0, \mathfrak{I}, \mathfrak{Y}, \mathfrak{Y})$. \square

5.2. Proof of Proposition 1.9 (i). Given $n \in \mathbb{Z}^2$ and $0 \leq t_2 \leq t_1$, define $\kappa_n(t_1, t_2)$ by

$$\begin{aligned} \kappa_n(t_1, t_2) &:= \mathbb{E}[\widehat{\mathfrak{I}}(n, t_1) \widehat{\mathfrak{I}}(-n, t_2)] = \int_{-\infty}^{t_2} e^{-(t_1-t')\langle n \rangle^2} \langle n \rangle^\alpha e^{-(t_2-t')\langle n \rangle^2} \langle n \rangle^\alpha dt' \\ &= \frac{e^{-(t_1-t_2)\langle n \rangle^2}}{2\langle n \rangle^{2(1-\alpha)}}. \end{aligned} \quad (5.13)$$

First, we prove that $\mathfrak{V}_N \in C(\mathbb{R}_+; \mathcal{C}^{-2\alpha-}(\mathbb{T}^2))$ with a uniform (in N) bound, almost surely. In view of (3.4) and (5.3), by repeating the computation in the proof of Lemma 3.1 (ii) (in particular (4.8) and (4.9)) and applying Lemma 2.3, we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathfrak{V}}_N(n, t)|^2] &\lesssim \sum_{n=n_1+n_2} \kappa_{n_1}(t, t) \kappa_{n_2}(t, t) \\ &\sim \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \lesssim \langle n \rangle^{-2+4\alpha}, \end{aligned} \quad (5.14)$$

provided that $0 < \alpha < \frac{1}{2}$. Since the time difference estimate follows from a slight modification, Lemma 2.6 implies that $\mathfrak{V}_N \in C(\mathbb{R}_+; \mathcal{C}^{-2\alpha-}(\mathbb{T}^2))$ almost surely. Moreover, a slight modification of the argument yields that $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ is almost surely a Cauchy sequence in $C(\mathbb{R}_+; \mathcal{C}^{-2\alpha-}(\mathbb{T}^2))$, thus converging to some limit \mathfrak{V} . Since the required modification is standard, we omit the details here.

Next, we show that when $\alpha \geq \frac{1}{2}$, we show that $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ for any $T > 0$ almost surely. From (5.14), we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathfrak{V}}_N(n, t)|^2] &\gtrsim \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ |n| \ll |n_1| \leq N}} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n - n_1 \rangle^{2(1-\alpha)}} \gtrsim \begin{cases} \log N, & \text{if } \alpha = \frac{1}{2} \\ N^{-2+4\alpha}, & \text{if } \alpha > \frac{1}{2} \end{cases} \\ &\rightarrow \infty \end{aligned} \quad (5.15)$$

as $N \rightarrow \infty$. Then, from Kolmogorov's three-series theorem, Kolmogorov's zero-one law, and Remark 4.3 with (5.15), we have

$$P\left(\lim_{N \rightarrow \infty} |\widehat{\mathfrak{V}}_N(n, t)| < \infty\right) = 0.$$

In particular, we obtain that $\{\mathfrak{V}_N\}_{N \in \mathbb{N}}$ forms a divergent sequence in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ almost surely for $\alpha \geq \frac{1}{2}$. This concludes the proof of Proposition 1.9 (i).

5.3. Proof of Proposition 1.9 (ii). First, we prove that $\Upsilon_N \in C(\mathbb{R}_+; \mathcal{C}^{2-2\alpha}(\mathbb{T}^2))$ with a uniform (in N) bound on each bounded time interval $[0, T]$, almost surely. As in Subsection 4.2, it suffices to show

$$\mathbb{E}[|\widehat{\Upsilon}_N(n, t)|^2] \lesssim_T \langle n \rangle^{-2-2(2-2\alpha)} \quad (5.16)$$

for any $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$.

We only consider $n \neq 0$ for simplicity. Proceeding as in (4.12), we have

$$\begin{aligned} & \mathbb{E}[|\widehat{\Upsilon}_N(n, t)|^2] \\ &= \frac{1}{\pi^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \int_0^t e^{-(t-t_1)\langle n \rangle^2} \int_0^{t_1} e^{-(t-t_2)\langle n \rangle^2} \kappa_{n_1}(t_1, t_2) \kappa_{n_2}(t_1, t_2) dt_2 dt_1 \\ & \quad + \frac{1}{2\pi^2} \cdot \mathbf{1}_{n \in 2\mathbb{Z}^2 \setminus \{0\}} \int_0^t e^{-(t-t_1)\langle n \rangle^2} \int_0^{t_1} e^{-(t-t_2)\langle n \rangle^2} \mathbb{E}\left[\widehat{\Upsilon}_N\left(\frac{n}{2}, t_1\right)^2 \overline{\widehat{\Upsilon}_N\left(\frac{n}{2}, t_2\right)^2}\right] dt_2 dt_1 \\ &=: \text{I}(n, t) + \text{II}(n, t), \end{aligned} \quad (5.17)$$

where $\kappa_{n_j}(t_1, t_2)$ is as in (5.13). From (5.13), we see that the contribution from $\text{II}(n, t)$ satisfies (5.16). Hence, we focus on $\text{I}(n, t)$. By (5.17) and (5.13), we have

$$\begin{aligned} \text{I}(n, t) &\sim \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \\ &\quad \times e^{-2t\langle n \rangle^2} \int_0^t e^{t_1(\langle n \rangle^2 - \langle n_1 \rangle^2 - \langle n_2 \rangle^2)} \int_0^{t_1} e^{t_2(\langle n \rangle^2 + \langle n_1 \rangle^2 + \langle n_2 \rangle^2)} dt_2 dt_1 \\ &= \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2 \\ |n_1|, |n_2| \leq N}} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \\ &\quad \times \frac{1}{\langle n \rangle^2 + \langle n_1 \rangle^2 + \langle n_2 \rangle^2} \left(\frac{1 - e^{-2t\langle n \rangle^2}}{\langle n \rangle^2} - e^{-2t\langle n \rangle^2} \frac{1 - e^{t(\langle n \rangle^2 - \langle n_1 \rangle^2 - \langle n_2 \rangle^2)}}{\langle n_1 \rangle^2 + \langle n_2 \rangle^2 - \langle n \rangle^2} \right) \\ &\lesssim \frac{1}{\langle n \rangle^2} \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{1}{\langle n_1 \rangle^{2(1-\alpha)} \langle n_2 \rangle^{2(1-\alpha)}} \frac{1}{\langle n \rangle^2 + \langle n_1 \rangle^2 + \langle n_2 \rangle^2}. \end{aligned} \quad (5.18)$$

By separately estimating the contributions from $|n_1| \sim |n_2| \gg |n|$ and $|n_1| \sim |n| \gtrsim |n_2|$ (or $|n_2| \sim |n| \gtrsim |n_1|$) with Lemma 2.3, we see that the contribution from $\text{I}(n, t)$ also satisfies (5.16) for $0 < \alpha < 1$. This proves (5.16).

Next, we show that when $\alpha \geq 1$, $\{\Upsilon_N\}_{N \in \mathbb{N}}$ does not converge in $C([0, T]; \mathcal{D}'(\mathbb{T}^2))$ for any $T > 0$ almost surely. From Remark 4.3, it suffices to show that

$$\lim_{N \rightarrow \infty} \mathbb{E}[|\Upsilon_N(n, t)|^2] = \infty \quad (5.19)$$

for $\alpha \geq 1$ under an appropriate assumption on $t > 0$.

Since the second term $\text{II}(n, t)$ in (5.17) does not involve any summation, it is finite. From (5.18), it is easy to see that the contribution to $\text{I}(n, t)$ from $|n| \sim \max(|n_1|, |n_2|)$ is finite. Indeed, assuming $|n_1| \lesssim |n| \sim |n_2|$ without loss of generality, the contribution

to (5.18) from this case is bounded by

$$\frac{1}{\langle n \rangle^{6-2\alpha}} \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ |n_1| \lesssim |n|}} \frac{1}{\langle n_1 \rangle^{2-2\alpha}} \lesssim \langle n \rangle^{-6+4\alpha}.$$

It remains to estimate $I(n, t)$ under the constraint $|n| \ll |n_1| \sim |n_2|$. When $t \gg |n|^{-2}$, the contribution to $I(n, t)$ from this case is bounded below by

$$\begin{aligned} \frac{1}{\langle n \rangle^2} \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ |n| \ll |n_1| \leq N}} \frac{1}{\langle n_1 \rangle^{6-4\alpha}} &\gtrsim_n \begin{cases} \log N, & \text{if } \alpha = 1 \\ N^{-4+4\alpha}, & \text{if } \alpha > 1 \end{cases} \\ &\longrightarrow \infty \end{aligned}$$

as $N \rightarrow \infty$. This proves (5.19) for $t \gg |n|^{-2}$, when $\alpha \geq 1$.

5.4. Proof of Lemma 5.2. As in Subsection 4.3, it suffices to show

$$\mathbb{E}[|\mathcal{F}[\Upsilon_N \ominus \mathfrak{I}_N](n, t)|^2] \lesssim_T \langle n \rangle^{-2-2(2-3\alpha)} \quad (5.20)$$

for $n \in \mathbb{Z}^2$ and $0 \leq t \leq T$, uniformly in $N \in \mathbb{N}$. As in (4.45), we decompose $\Upsilon_N \ominus \mathfrak{I}_N$ into two parts:

$$\begin{aligned} \mathcal{F}[\Upsilon_N \ominus \mathfrak{I}_N](n, t) &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_1+n_2| \sim |n_3| \\ n_1+n_2 \neq 0}} \int_0^t e^{-(t-t')\langle n_1+n_2 \rangle^2} \widehat{\mathfrak{I}}_N(n_1, t') \widehat{\mathfrak{I}}_N(n_2, t') dt' \cdot \widehat{\mathfrak{I}}_N(n_3, t) \\ &\quad + \frac{1}{4\pi^2} \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ |n_1| \leq N}} \mathbf{1}_{|n| \sim 1} \int_0^t e^{-(t-t')} \cdot (|\widehat{\mathfrak{I}}(n_1, t')|^2 - \kappa_{n_1}(t')) dt' \cdot \widehat{\mathfrak{I}}_N(n, t) \\ &=: \widehat{\mathcal{R}}_1(n, t) + \widehat{\mathcal{R}}_2(n, t). \end{aligned}$$

Moreover, we decompose \mathcal{R}_1 as

$$\begin{aligned} \widehat{\mathcal{R}}_1(n, t) &= \frac{1}{4\pi^2} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_1+n_2| \sim |n_3| \\ (n_1+n_2)(n_2+n_3)(n_3+n_1) \neq 0}} \int_0^t e^{-(t-t')\langle n_1+n_2 \rangle^2} \widehat{\mathfrak{I}}_N(n_1, t') \widehat{\mathfrak{I}}_N(n_2, t') dt' \cdot \widehat{\mathfrak{I}}_N(n_3, t) \\ &\quad + \frac{1}{2\pi^2} \int_0^t \widehat{\mathfrak{I}}_N(n, t') \left[\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} e^{-(t-t')\langle n+n_2 \rangle^2} \right. \\ &\quad \quad \quad \left. \times \left(\widehat{\mathfrak{I}}(n_2, t') \widehat{\mathfrak{I}}(-n_2, t) - \kappa_{n_2}(t, t') \right) \right] dt' \\ &\quad + \frac{1}{2\pi^2} \int_0^t \widehat{\mathfrak{I}}_N(n, t') \left[\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2| \neq 0 \\ |n_2| \leq N}} e^{-(t-t')\langle n+n_2 \rangle^2} \kappa_{n_2}(t, t') \right] dt' \\ &\quad - \frac{1}{4\pi^2} \cdot \mathbf{1}_{n \neq 0} \int_0^t e^{-(t-t')\langle 2n \rangle^2} (\widehat{\mathfrak{I}}_N(n, t'))^2 dt' \cdot \widehat{\mathfrak{I}}_N(-n, t) \end{aligned}$$

$$=: \widehat{\mathcal{R}}_{11}(n, t) + \widehat{\mathcal{R}}_{12}(n, t) + \widehat{\mathcal{R}}_{13}(n, t) + \widehat{\mathcal{R}}_{14}(n, t). \quad (5.21)$$

Proceeding as in the proof of Proposition 3.2, we can easily show that $\widehat{\mathcal{R}}_{11}$, $\widehat{\mathcal{R}}_{12}$, and $\widehat{\mathcal{R}}_{14}$ satisfy (5.20).

It remains to consider $\widehat{\mathcal{R}}_{13}$. Under the constraint $|n_2| \sim |n + n_2|$, we have $|n_2| \gtrsim |n|$. Then, from (5.21) with (5.13), we have

$$\begin{aligned} \mathbb{E}[|\widehat{\mathcal{R}}_{13}(n, t)|^2] &\lesssim_T \frac{1}{\langle n \rangle^{2-2\alpha}} \sum_{\substack{n_2 \in \mathbb{Z}^2 \\ |n_2| \sim |n+n_2|}} \frac{1}{\langle n + n_2 \rangle^2 \langle n_2 \rangle^{2-2\alpha}} \\ &\quad \times \sum_{\substack{n'_2 \in \mathbb{Z}^2 \\ |n'_2| \sim |n+n'_2|}} \frac{1}{\langle n + n'_2 \rangle^2 \langle n'_2 \rangle^{2-2\alpha}} \\ &\lesssim \langle n \rangle^{-6+6\alpha} \end{aligned}$$

for $0 < \alpha < 1$, verifying (5.20).

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